Mitigating Global Warming: A Real Options Approach

Abstract

Mitigation and adaptation represent two solutions to the issue of global warming. While mitigation aims at reducing CO_2 emissions and preventing climate change, adaptation encompasses a broad scope of techniques used to reduce the impacts of climate change once they have occurred. Both have direct costs on a country's Gross Domestic Product, but costs also arise from temperature increases due to inaction. This paper introduces a tipping point in a real options model and analyzes optimal investment choices in mitigation and their timing.
Keywords: Adaptation, Mitigation, Real Options, Delay, Tipping Point, Climate Change, CO_2, Gross Domestic Product

1 Introduction

Climate change has become increasingly important in political discussions. The Intergovernmental Panel for Climate Change (IPCC) has expressed strong concerns about the eventual consequences for the planet and humanity if mean temperatures reach or increase above the $2°C$ threshold. Since this temperature increase seems inevitable at this point, given the CO_2 emission trend of past years, the IPCC is calling for rapid efforts to prevent further warming, via *mitigation*, and to reduce the effects of already rising temperatures on natural and social systems, via *adaptation* (IPCC, 2014). Indeed, climate change has huge potential negative effects. Lower food supplies, water shortages, droughts, and increased health problems are among the consequences of high CO_2 concentrations that negatively influence production and consumption, which in turn impact current and future economic growth. The situation is already more critical than expected, and negative feedback effects are imminent (IPCC, 2014). Other potential effects are catastrophic, both in terms of system dynamics and in the common meaning of the world. The potential collapse of the Atlantic thermohaline circulation and its effects on the lives of millions of people is a clear example of such abrupt changes (Huber and Knutti, 2012).

To take these potential effects into account, this article models the impact of climate change and the possible occurrence of a catastrophic event on global welfare. The possibility of catastrophic events is widely acknowledged in the literature, and their implications have been investigated at both theoretical and empirical levels (see next section). Catastrophic events occur when the state of the climate reaches a tipping point -the threshold- with strong feedbacks that trigger one or several events. Such catastrophic events could include the interruption of the thermohaline circulation, massive methane releases, or the melting of ice caps causing a rise in sea level. In this paper, we define a catastrophic event as an irreversible disruption having a dramatic negative impact on humanity. If the catastrophic sequence of events is triggered, even a return to pre-industrial conditions will only allow the ice caps or the methane sinks to reconstitute themselves over such a long period that their loss may be considered irreversible for the purposes of human society. The change in climate regime and the new conditions prevailing over the planet will thus be established irreversibly. Although the human species would not be wiped out, the costs would be high enough -and the subsequent conditions of human activity uncertain enough- that it is justified to model this catastrophe as a long-lasting collapse of the Gross Domestic Product (GDP) and as an interruption of the dynamic optimization problem addressed by our model for the period preceding the catastrophe. We assume that the climate state defining a tipping point can be modelled as an atmospheric temperature threshold. Our paper further considers an aspect neglected in climate and economic modelling: catastrophes are likely to be triggered only if temperatures stay above some threshold for a certain time (Lenton et al., 2008). This time window has to be given particular attention. Short periods above the temperature threshold would not lead to any drastic departure from the continuous pattern of damage associated with temperature while a long period above the threshold temperature would trigger a catastrophe.

The above assumptions imply that the catastrophe is certain not to happen in the immediate future as long as temperature stays below some threshold level. However, the likelihood of a catastrophe occurring within a given future period increases as temperature rises, since the rise means that the threshold becomes more likely to be reached and also exceeded by the process for the duration of the *time window*. Furthermore, a long-lasting business-as-usual policy will lead to a catastrophe. Consequently, the decision maker must monitor the temperature process and decide whether or not to devote resources to slow down or reverse the rise in temperature. This is the mitigation decision. Mitigation has been studied in a number of ways that we discuss briefly in the next section. We model it as a once-and-for-all irreversible decision to start spending some endogenous proportion of GDP on it after some optimally chosen temperature level has been reached. This determines a reduction in emissions and thus a modification of the temperature process, which is stochastic in our setting. While this is a typical real options setup, its solution is not conventional and involves a methodological contribution outlined in the text and precisely described in the appendix.

Adaptation is different from mitigation. First mitigation is a pure global public good while adaptation involves actions that are either private or whose public dimension is much less pronounced. The decision by an individual to move to safer grounds is largely private. Protective dikes are public goods, but only locally, and public institutions deal much better locally than globally for the provision of public goods. With a climate treaty, for example, free rider problems appear. For this reason, we treat adaptation as exogenously determined within the model, without any intervention of the decision maker, while we treat mitigation as a planning decision. The decision maker optimizes mitigation for an economy whose GDP already incorporates the consequences of decentralized adaptation. The second important difference between adaptation and mitigation is that adaptation does not affect the temperature process that determines climate. As a result, adaptation

has no effect on the probability of occurrence of a catastrophe. We further assume that, when the catastrophe occurs, adaptation measures taken prior to the event are without effect on the consequences of the catastrophe, which is that GDP equals zero as of this date. This is because the damages are different in nature from those resulting from a progressive change in climate and are also much more difficult to envisage. Consequently, we treat adaptation as a decentralized activity affecting welfare before the possible climate catastrophe but without any impact on its consequences. Given the path of GDP, net of the impact of adaptation, the decision maker optimizes the additional welfare impact of mitigation while considering its effect on the probability of climatic catastrophe.
The questions that our research is trying to answer are the following: (i) What is the optimal percentage of Gross Domestic Product, net of adaptation expenditures, that a global decision maker should invest in climate change mitigation efforts each year in order to maximize the expected discounted sum of future GDPs, given that higher temperatures imply higher financial burdens? (ii) What is the optimal mean temperature that should trigger investment in mitigation? (iii) Must mitigation expenditures be higher than those for adaptation, or vice versa? And finally, (iv) how do investments in mitigation affect the probability of the occurrence of catastrophic events? We will provide detailed answers to these questions in Section 5.

The article is structured as follows. The current state of the literature on GDP impacts of climate change will be discussed in Section 2 while the model will be presented and explained in detail in Section 3. Section 4 will present the dynamic optimization of the model, Section 5 will show the numerical results obtained, and Section 6 will draw some conclusions.

2 Literature Review

A number of issues on climate change are addressed in the economic literature. These include the cost of climate change, the potential for mitigation and adaption, and the instruments that must be mobilized as well as the timing of action. The impacts are identified in terms of growth in GDP, food supply, or the stock of man-made or natural capital. Empirical assessments differ widely, but there is a broad consensus that impacts are unevenly distributed across world regions.[1] Another area of consensus is that climate

[1]The literature investigating the economic impacts of climate change, and the need for mitigation and/or adaptation measures, has usually focused on specific areas and sectors, in particular on agriculture and in general on the future availability of food supplies. Few authors have operated in a global setting. Fischer et al. (2005), combining a bio-physical analysis with an economic one, studied the interactions between climate change and different development paths. The importance of their analysis is based on the fact that they are able to distinguish between impacts on developed and developing countries. The results suggest that climate change will worsen the gap between these two groups of countries, in terms of production and consumption possibilities. For Fischer et al. (2005), adaptation in agricultural techniques is the key to limiting the impacts of climate change on crops. Rosenzweig and Parry (1994) studied the effects of climate change on food supply. Their research showed that adaptation at a local-farm level is insufficient. Action, in the form of mitigation, is needed at a global, and thus more incisive, level. These conclusions were also drawn by Parry et al. (2004), who specifically considered the impacts of climate change on food supply with different socio-economic scenarios under the Intergovernmental Panel on Climate Change, Special Report on Emissions Scenarios (Nakićenović and IPCC, Working Group III., 2000). While their results depend heavily on the effects of CO_2 concentration on agriculture yields, which

change is an immense challenge for economic institutions. First, and despite a tempering note by Battaglini et al. (2014), this is because climate change is the biggest instance of the tragedy of the commons ever recorded (Stern, 2007; Stavins, 2011). As such, it cannot be addressed without some interference with the decentralized operation of markets. Second, climate change is the first instance of the tragedy of the commons occurring at a truly global scale. It is not likely to be solved by the methods that societies have developed at local and regional levels to deal with similar problems at smaller scales. A theoretical literature initiated by Barrett (see, e.g., Barrett, 2005; 2013) analyzes the difficulties involved in reaching international agreements in that context. As a result, much of the literature is normative, and our paper also falls into this category.

A substantial part of the economic climate change literature consists of integrated assessment models (IAM). Although some have the appearance of positive analyses, their conclusions are invariably used to fuel debates over normative issues. Economic models of climate change and their outcomes have been investigated by Nordhaus and Boyer (2003) and Tol (2002a). Nordhaus and Boyer (2003) developed a model, called RICE for Regional Integrated model of Climate and Economy, which is an improvement of the famous DICE model (Dynamic Integrated model of Climate and Economy; Nordhaus, 1992).[2]

One of the most important studies about the effects of climate change on world GDP is that of (Stern, 2007). This author calculates the monetary impacts of inaction (or insufficient action) on the global economy. Stern (2007) found that due to inaction, the world may lose up to 5% of its aggregate GDP each year. If all the possible risks are taken into account, as in a worst-case scenario, costs on world GDP could add to 20% or even more. These costs are very high compared to what the author calculated as the amount needed to combat climate change, i.e., about 1% of world GDP if carbon has to be stabilized at around 550 ppm. Similar results, though slightly smaller, are found in IEA (2006). A recent study by Fundacion DARA Internacional and Climate Vulnerable Forum (2012) on the monetary impacts of climate change on world GDP found that about

are unknown, Parry et al. (2004) infer that the world, as one entity, will be nevertheless able to feed itself, since the diminished production in developing countries will probably be counterbalanced by an increased production in developed countries. However, this result does not justify inaction, since consequences of these inequalities at regional and local levels may become socially and economically devastating for less developed countries. Fankhauser (1997) estimated costs and benefits of climate change and how these impact economies in six regions, namely the former Soviet Union, China, the United States of America, the European Union, other OECD countries, and other non-OECD countries. That author found that climate change is most likely to cause a loss of 1.5% of the world GDP per year, while reaching the threshold of $2.5°C$ above pre-industrial temperature levels.

[2]The RICE model is able to predict the economic impacts of climate change in different regions. Like other important literature, results show that developed regions would, on average, profit from an increase in global mean temperatures, while the impact on developing countries would be the opposite. However, the two effects are not of the same magnitude, since climate change affects the poorest areas with much more devastating outcomes compared to what richer communities would experience. Tol (2002a) finds that the impacts of changing climate on GDP are extremely model-dependent, since they can be positive, negative, or non-existing according to how prices are taken into account. In general, however, it is evident how impacts have different consequences depending on the country or group of countries under analysis, whether developed (OECD, Middle East, China) or non OECD. Similar results were obtained by Tol (2002b), where dynamic estimates were introduced.

3.2% of world GDP by 2030 (or 1.2 trillion a year) are at risk because of climate change and because of the inaction of governments around the world. The Stern review relies on various information and methodologies, especially those of IAMs.

IAMs have been harshly criticized for their lack of objectivity and transparency in policy applications.[3] Pindyck (2015) and others argue in favour of simple pedagogical models able to enlighten decisions but that certainly must not be relied on for accurate answers. A variety of models may claim to fall into that category. A brief review not only shows their variety and richness, but helps identify and justify the climatic and economic features that we choose to emphasize in this paper. Golosov et al. (2014) developed a dynamic stochastic general equilibrium (DSGE) model that allows the identification of the optimal carbon tax, or, equivalently, the marginal externality damage of emissions.[4] Bretschger and Vinogradova (2014) model an economy in which global warming causes stochastic climate shocks that negatively impact the capital stock. They found an optimal flow of emission abatement that is able to reduce climate shocks.[5] Mitigation and adaptation represent the core of the analysis by Bahn et al. (2012). These authors found that investments in mitigation highly depend on the effectiveness of adaptation measures. In fact, highly effective adaptation measures may optimally hinder -or completely cancel- any potential medium-to-long term effort in mitigation.[6] can be suboptimal due to uncertainty about the future (de Bruin and Dellink, 2011). However, while preventive adaptation may include strong delays before being effective, as happens with mitigation (Bahn et al., 2012), reactive adaptation reduces uncertainty and delivers results more quickly, as pointed out by Parry et al. (2009). Prieur et al. (2011) and Amigues and Moreaux (2013) introduced a threshold catastrophic temperature as the key element of an economic climate change model where the catastrophe causes infinitely large damage. While they used a dynamic but non-stochastic framework, Tsur and Zemel (2008) also consider the possibility of a catastrophic climate event in a stochastic environment. However, the random occurrence of the catastrophe is not directly linked to temperature or CO_2 thresholds; it is governed by a Poisson law, where the parameter increases with cumulative emissions. While they differ in their treatment of uncertainty, the above papers highlight the importance of catastrophes. Pindyck (2015) claims that "what really matters [for the social cost of climate change] is the likelihood and possible impact of a catastrophic climate outcome: a much larger-than-expected temperature increase and/or

[3]Pindyck (2015) argues that "...Because the modeller has so much freedom in choosing functional forms, parameter values, and other inputs, the model can be used to obtain almost any result one desires, and thereby legitimize what is essentially a subjective opinion about climate policy."

[4]The authors find that the damage is proportional to the current GDP and that the degree of proportionality is only dependent on the discount rate, on the elasticity of damage, and on the structure of carbon depreciation in the atmosphere. Interestingly, important elements such as consumption, population dynamics, technological paths, and CO2 concentration in the atmosphere have no influence on the damage caused by emissions. In addition, they find that the optimal carbon tax should be higher than the median tax estimated by the literature.

[5]The optimal flow spent on abatement exhibits a constant growth rate and is an increasing function of the intensity of the environmental damage. They suggest that a world with uncertainty requires more stringent climate policies that a world without.

[6] If a very costly catastrophe will occur with some probability when a threshold temperature is reached and not subjected to adaptation but only to mitigation, then adaptation increases the probability of such catastrophe as it takes resources away from mitigation.

a much larger-than-expected reduction in GDP caused by even a moderate temperature increase. IAMs, however, simply cannot account for catastrophic outcomes". In our paper, we model a climate catastrophe as an irreversible event of such magnitude and with such manifestations that it amounts to the end of society as we knew it before the catastrophe, with no basis to conceive of the ensuing area. We model this as calling an end to the optimization period. Although this is by design an extreme representation of a climate catastrophe, it is not without scientific basis. Dakos et al. (2008) and Lenton et al. (2008) find that a deviation from a threshold temperature sustained over time is capable of inducing dramatic changes to the environment. Lenton et al. (2008) identified several policy-relevant tipping elements, i.e., events or climate states that could keep the temperature process above a certain threshold for a long time window.

Researchers differ widely in the way they have modelled catastrophes and their consequences. Baranzini et al. (2003) model an environmental catastrophe incorporating negative jumps in the stochastic process corresponding to the net benefits associated with abatement policies. They find that policies, that would be optimal under traditional cost-benefit analyses, should now be rejected and delayed to a future date. Moreover, their results imply that the possibility of climate catastrophes increases the probability of implementing GHG abatement policies. Lemoine and Traeger (2014) investigated the welfare costs of a tipping point, finding that a sufficiently high carbon tax is necessary to mitigate abrupt climate shifts and that such a tax is capable of reducing peak temperatures by as much as $0.5°C$. Following the set-up of Naevdal (2006), Naevdal and Oppenheimer (2007) deal with the trigger of an environmental catastrophe, i.e., the interruption of thermohaline circulation, which would occur if "the temperature or rate of temperature change exceed certain [unknown] thresholds". The authors distinguished two unknown thresholds that trigger the collapse of thermohaline circulation. One is related to the rate of temperature increase and one is related to the temperature level itself. The decision maker chooses upper bounds for these processes that he is not willing to cross. In other words, these upper bounds trigger his decision to mitigate global warming.[7] Similarly, Keller et al. (2004) studied the effects of an unknown threshold that causes the interruption of thermohaline circulation. These optimal stopping models are similar in that respect to the real option model presented in this paper.[8] Weitzman (2007) found the probability of crossing a threshold temperature level higher than $8°C$ relative to pre-industrial level to be approximately $3\% - 4\%$; the negative consequences

[7]Naevdal and Oppenheimer (2007) found that the upper boundary of the rate of temperature increase is crossed in finite time while the upper boundary of the temperature process is crossed only as time goes to infinity.

[8]The authors found that increased uncertainty does not increase optimal abatement. The reason can be found in one (or both) of these conditions: (a) risk aversion is not the dominant nonlinearity in their model and (b) increased uncertainty does not decrease the variance of the per capita consumption. Thus, characteristics of the threshold and the learning process have a strong influence on the optimal abatement policy in the near-term. Similar results in a setting with an unknown (but reversible) catastrophic threshold are also analyzed by Brozović and Schlenker (2011), who found a non-monotonic relationship between precautionary behaviour and uncertainty. Higher uncertainty surrounding the natural system usually increases precautionary behaviour. However, when the risk becomes large enough, the behaviour of the decision maker becomes less precautionary because "precautionary reductions in pollutant will be too costly compared to the negligible expected reduction in the probability that the threshold is crossed".

of such a catastrophe are impossible to appraise, whether qualitatively or in magnitude.

Our model is strongly inspired by the literature on tipping points triggering catastrophic natural events. That is, the catastrophe is certain not to happen as long as the threshold is not reached, in contrast to models where a catastrophe is possible with some probability whatever the state (as when its occurrence obeys a Poisson law). Similar to Amigues and Moreaux (2013), we model the catastrophe as a dramatic event of such magnitude that there is no need or possibility to model -let alone manage- the post-catastrophe regime, as in Lemoine and Traeger (2014) and the controlled IAMs that they discuss. We do so in a stochastic environment. To avoid unrealistic outcomes, where the catastrophe occurs with certainty as soon as a known threshold is reached, many authors have assumed that the threshold is unknown, implying that learning about the threshold may occur, i.e., if some state is reached and no catastrophe occurs, one knows that the threshold must be higher. The economy is then safe if it remains at or below the state already reached. Our setting differs in that the (temperature) threshold is assumed to be known, but the catastrophe occurs only if enough time passes above that threshold. Since the process is stochastic, the catastrophe is uncertain even when the threshold is exceeded, but society is obviously taking risks if it allows that to happen. We believe that this is a more realistic way to model the scientific evidence described in Lenton et al. (2008). Global temperature levels are a good example, since a prolonged period above a certain temperature level is needed for a natural catastrophe such as the interruption of thermohaline circulation or melting of the Western Antarctic Ice Sheet to occur.

Perhaps the most important characteristic that distinguishes economic from other dynamic stochastic climate models is when the former seek to optimize some policy variable. Remarks such as those of Weyant (2008) about the Stern review -that climate policy should not be taken as a deterministic "one shot" benefit-cost analysis but as a problem of sequential decision-making under uncertainty- are to be taken seriously. However, the optimal stochastic control of one or several variables over time[9] raises issues that are not only computational and may justify the consideration of "one shot" decisions. The environmental real option approach is based on the premise that environmental policies are irreversible due to institutional or other constraints, so they are best modelled as once-and-for-all (or long-term) decisions (Pindyck, 2000; Insley, 2002; Kassar and Lasserre, 2004). This is perhaps most obvious if we think that climate problems may have to be solved by treaties (Barrett, 2013). In such cases, and many others, environmental policy decisions are costly to reach and to characterize, so they take the form of a simple policy decision that is irreversible and requires dedicated resources, whether they are dollars or political capital. The timing of the policy and the magnitude of the engagement must be chosen optimally. As described by Pindyck (2000), these decisions involve two kinds of irreversibilities that work in opposite directions. First, an environmental policy imposes sunk costs on society, and political constraints may make the policy itself difficult to reverse. Second, environmental damage can be partially or totally irreversible. For example, increases in GHG concentrations are long lasting, and the damage to ecosystems

[9]Examples are the optimal carbon tax in an stochastic dynamic general equilibrium SDGE model of Golosov et al. (2014), the model with tipping points of Lemoine and Traeger (2014), or the optimal flow of carbon abatement in Bretschger and Vinogradova (2014).

from higher global temperatures can be permanent. Thus, adopting a policy now rather than waiting has a sunk benefit -a negative opportunity cost- that biases traditional cost-benefit analysis against policy adoption. Our model examines a similar trade-off. We assume that the decision maker knows the tipping point temperature and that he has to make an optimal decision, in terms of when to invest in mitigation, by choosing an optimal temperature threshold, and in terms of how much to invest, by choosing an optimal fraction of GDP to devote to mitigation.

3 The Model

The average global surface temperature (Hasselmann, 1976; Kaerner, 1996; Lawrence and Ruzmaikin, 1998; Eby et al., 2009, 2012) and world GDP dynamics (Brock and Mirman, 1972) can be respectively modelled by a time component plus a random part, driven by white noise and a volatility parameter, i.e.,

$$dC_t = \begin{cases} adt + \beta dW_t & \text{for} \quad t < T_L + \Delta T(k, L) \\ a(k)dt + \beta dW_t & \text{else} \end{cases} \tag{1}$$

and

$$\frac{dV_t}{V_t} = \mu dt + \sigma dB_t \tag{2}$$

where $\{W_t, t \geq 0\}$ and $\{B_t, t \geq 0\}$ are two independent Brownian motions[10] under the physical probability $\mathbb{P}$ and where the drift parameters, a and μ, and the volatility parameters, β and σ, are constant and positive. In particular, $a > 0$, the drift of the temperature process, explicitly models the global warming effect we are experiencing today. We will also assume that the discount rate r is constant and positive. It is worthwhile mentioning that there is no guarantee that the temperature process C will remain positive with probability one. However, given our initial set of parameters, it is highly likely. The GDP process V, expressed in dollars, is by definition positive.
The temperature process evolves in two phases, a "before mitigation" and an "after mitigation" phase beginning at time T_L, as soon as the threshold L is reached. The impact of such a mitigation strategy on the temperature process will start with a delay, i.e. , at time $T_L + \Delta T(k, L)$. There is also an autonomous GDP process and a *net* GDP process; the latter is a function of both temperature excess from the pre-industrial temperature, $C_P = 14°C$ and the autonomous GDP process V_t. To be more specific, climate change causes a flow of day-to-day costs over time, and these costs can be viewed as levies from GDP as time goes by. Human adaptation efforts can reduce their immediate impacts to some extent, but not suppress them. Consequently, we introduce the disposable GDP, $DGDP_t$, as the GDP, V_t, net of the day-to-day costs of climate change as moderated by adaptation efforts:

$$DGDP_t = V_t e^{-\rho|C_t - C_P|} \tag{3}$$

where $C_P = 14°C$ describes the global average temperature level prior to industrialization, in the absence of man-made pollution, and where $\rho > 0$ is a parameter reflecting

[10] In an arithmetic Brownian motion setting, the drift a and the volatility β are both expressed in degrees Celsius.

the impact of the temperature gap and its measurement units. Note that this functional form implies strong convexity with respect to $C_t - C_P$, meaning that the effect of inaction is accentuated if the temperature process C and the temperature level prior to industrialization, C_P, diverge. The higher the difference between C_t and C_P, the more accentuated its impact on adaptation costs and therefore on the disposable GDP.

A global environmental catastrophe will occur if the temperature remains without interruption above a given temperature level L_1 over a period of D units of time. This specification finds its justification in a vast literature on tipping points and abrupt climate change, which is reviewed with a focus on policy implications by Lenton et al. (2008). Abrupt climate change occurs "when the climate system is forced to cross some threshold, triggering a transition to a new state at a rate determined by the climate system itself and faster than the cause" (p. 1786). In fact, deviations above the tipping point L_1, sustained over time (for D units of time), are capable of creating serious negative effects on the environment. According to Hansen et al. (2008),

> Paleoclimate data and ongoing global changes indicate that 'slow' climate feedback processes not included in most climate models, such as ice sheet disintegration, vegetation migration, and GHG release from soils, tundra or ocean sediments, may begin to come into play on time scales as short as centuries or less. (p. 217)

Indeed, as these authors argue, if the overshoot of the appropriate long-run CO_2 target is not brief, there is a high probability of seeding irreversible catastrophic effects.[11] The catastrophe will therefore not occur the first time that the temperature reaches the critical level L_1, but only if it remains above this critical level without interruption for a given period of time. The full impact of global warming on possible climate catastrophes therefore requires a given time window. As soon as the catastrophe occurs, the GDP is approximated by zero and is assumed to remain at this level as of this date. This is a specific feature of this model as compared with others. The real options setting allows the joint determination of the optimal temperature level L^* at which mitigation should start to be implemented and the optimal level k^* of this investment. Optimality means that these two values are endogenously specified in order to maximize the expected sum of discounted GDPs between the current time and the date of the catastrophe.[12]

[11] Hansen et al. (2008) argue that 'If humanity wishes to preserve a planet similar to that on which civilization developed and to which life on Earth is adapted, paleoclimate evidence and ongoing climate change suggest that CO_2 will need to be reduced from its current 385 ppm to at most 350 ppm.(...) If the present overshoot of this target CO_2 is not brief, there is a possibility of seeding irreversible catastrophic effects.'

While the target of 350 ppm is low when compared with other targets that are considered reasonable, in particular some proposed by the IPCC, the idea that a long overshoot will trigger a catastrophe that might have not occurred with a brief incursion in the non-sustainable zone, appears very reasonable.

[12] Similar situations have been studied in finance with Parisian options; the delay is called Parisian time. For the mathematical specification of Parisian time and other variables, please refer to Appendix A.

3.1 Adaptation and Mitigation

Mitigation and adaptation are modelled as follows: the mitigation option as the flow of investments k as of an optimal date, capable of decreasing the drift of the temperature process, and adaptation as implied by the constant ρ in Eq. 3. The decision maker's objective function is to maximize the expected sum of disposable GDP over k and over the temperature threshold L that triggers mitigation.

Adaptation The higher ρ the bigger the net day-to-day impact of climate change on disposable GDP. A value of $\rho = 0$ means that there is no impact of temperature change on GDP, hence no adaptation efforts, so that $DGDP_t = V_t$. When ρ is strictly positive, disposable GDP would depart from V_t as the temperature gap $C_t - C_P$ rises. As temperature increases, an increasing proportion of GDP is lost to the day-to-day costs of climate change.

Adaptation efforts range from entirely private (changing residence) to partially public (building levies for local protection), as opposed to mitigation, which is a pure public good at the global scale. It is thus reasonable to assume that ρ is determined by market mechanisms and local institutions that function efficiently, whereas mitigation decisions have to be studied as a decision maker's problem. Adaptation efforts have no effect on climate dynamics, they only affect the way current temperature translates into current disposable GDP, and they only do so while climate dynamics remain in the current climate regime. If temperature rises to such a level that the climate dynamic system undergoes some catastrophic change, previous adaptation efforts will not have any impact on the magnitude of the catastrophe.

Mitigation Mitigation aims to slow down climate dynamics, as represented by Eq. 1. After an optimal date (denoted T_L in the model[13]), an entity, such as an international organization, chooses to devote a proportion $ke^{-\delta(s-T_L)}$, with $s \geq T_L$, of world disposable GDP to activities or measures that reduce the rate of increase of the temperature process relative to some business-as-usual trajectory. This proportion may be constant if $\delta = 0$ or, if $\delta > 0$, it diminishes exponentially from its maximum k that occurs at the date T_L on which the mitigation decision is taken. When $\delta < 0$, then too much time has passed with inaction, and a mitigation effort increasing over time is needed. Given the difficulties surrounding the mitigation decision process, the decision to slow down the process driving climate change should be viewed as being reached very rarely -we assume once at most- and as irreversible. For example, it may be interpreted in the model as a treaty whose features would be respected once the treaty is signed, although there might be a delay until these features are fully implemented and a delay until their effect is felt. For example, consider a decision that is implemented when the global temperature reaches some endogenous threshold level L, at date T_L. If a fraction $ke^{-\delta(t-T_L)}$ is spent for mitigation as of T_L, then the temperature process will be *modified* only after a given delay $\Delta T(k, L)$, i.e., the trend of the temperature will be set to $a(k)$ instead of remaining

[13] As shown later, T_L is the random time at which the temperature process reaches a predetermined level L, which triggers the flow of investments k in mitigation.

at a at time $T_L + \Delta T(k, L)$. The delay is defined as

$$\Delta T(k, L) = \frac{\theta}{V_{T_L} e^{-\rho(L-C_P)} k} \tag{4}$$

where T_L represents the date at which the decision to mitigate is taken, i.e., the first passage time of the temperature process at level L, and θ is a parameter that models the delay magnitude, which can be influenced by chemical and atmospheric factors, and also by elements such as type of mitigation, whose choice is not modelled here. The delay $\Delta T(k, L)$ takes into account the fact that the higher the starting disposable GDP available for expenditures in mitigation and the higher the fraction of disposable GDP actually spent, k, the quicker the effect on the temperature process. In addition, the wider the temperature gap $L - C_P$ at the time the mitigation decision is implemented, the longer the delay because the day-to-day costs of climate change, despite adaptation efforts, use up a portion of the disposable income otherwise available for mitigation. The function of $\Delta T(k, L)$ finds its rationale in important scientific literature (Friedlingstein et al., 2011), which showed that, despite a sudden drop in carbon emissions and due to strong inertia, it still takes (much) time for the temperature process to stabilize and eventually start decreasing. For additional information, please refer to Appendix C.

In addition to the delay required for mitigation expenditures to become effective, their initial size k, as a proportion of disposable capital, determines their impact on the temperature trend, which they reduce from a to $a\,(k)$ according to the formula[14]

$$a(k) = a - (a - \eta)\frac{k}{\alpha} \tag{5}$$

with

$$\lim_{k \to \alpha} a(k) = \eta \tag{6}$$

where η is a negative constant reflecting the self-regenerating capacity of the atmosphere, expressed here in terms of its effect on the temperature process; its determination will be described shortly. Eq. 6 indicates that when mitigation efforts are set to a proportion $k = \alpha \leq 1$ of $DGDP$, then the drift of the temperature process reaches η so that anthropogenic effects are eliminated and the self-regenerating capacity of the atmosphere becomes the sole non-stochastic factor affecting temperature changes. In other words, $\alpha \cdot 100\%$ of the GDP should be spent in order to eliminate these anthropogenic effects. Note that in this model, an $\alpha = a/(a - \eta) < 1$ is required in order to stabilize the temperature level ($a = 0$). While an $\alpha = 1$ corresponds to a pessimistic scenario, it is an indisputable ceiling that corresponds to our choice in this paper. Obviously, a smaller value of α could be used in our model. We will illustrate this possibility in Section 5.
Equation 5 can be rationalized as follows. Assume that Equation 1 is an approximation of

$$dC_t = (\eta C_t + f(N) + f(E_t))\, dt + \beta dW_t \tag{7}$$

where $f(\cdot)$ represents a function that models the impacts of natural emissions N and of anthropogenic emissions E_t on the temperature process per unit of time, and ηC_t is the drop in temperature induced by the gross (before emissions) self regeneration of

[14] Since a is independent of δ at this stage, we will write $a(k)$ instead of a of $a(k, \delta)$.

the atmosphere, when temperature is C_t; η is a negative parameter to be determined. $f(N)$ is assumed to remain constant over the industrial period while $f(E_t)$ is null at the beginning of industrialization and positive thereafter. It is assumed that the drift of the temperature process was zero before the industrial period, since natural emissions on average offset self regeneration; hence

$$f(N) = -\eta C_P \tag{8}$$

where $C_P = 14.0°C$ represents the average global pre-industrial temperature. Substituting into Eq. 7 yields:

$$dC_t = \left(\eta \left(C_t - C_P\right) + f(E_t)\right) dt + \beta dW_t. \tag{9}$$

Since $\eta < 0$, Eq. 9 models the dynamics of a mean reverting process. In particular, if $C_t = C_P$, the drift reduces to $f(E_t)$.
In this paper, Eq. 1 is used instead of the complex Eq. 9 to model the temperature process. The drift a corresponds to:

$$a \cong \eta \left(C_0 - C_P\right) + f(E_0) \tag{10}$$

The negative parameter η may be approximated as follows. The self-regenerating capacity of the atmosphere is often defined as the natural rate of resorption of the CO_2 stock (Hansen et al., 2008; Archer et al., 2009); estimates vary widely. The assumption that the natural rate of resorption of the CO_2 stock is constant and that 25% of emitted CO_2 is still in the atmosphere after two centuries (Friedlingstein et al., 2011) implies a decay rate of 0.1% per year. In terms of temperature process, we recall that the atmosphere was in stationary equilibrium at a temperature of $C_P = 14.0°C$ during the pre-industrial period. Suppose that this equilibrium is disturbed by the sudden emission of a quantity of carbon that instantaneously raises temperatures by one degree. That carbon will still be in the atmosphere after two centuries, so that the gap in temperature away from the stationary equilibrium will vanish at the rate of 0.1% per year because of the natural decay of CO_2. Thus the influence of natural carbon decay on temperature is $0.001 \left(C_t - C_P\right)$ and

$$\eta = -0.1\% \tag{11}$$

At dates before $T_L + \Delta T(k, L)$, the dynamics of the temperature process are given by (1). By spending a proportion $ke^{-\delta(t-T_L)}$ of disposable GDP, as of T_L it is possible to reduce anthropogenic emissions, thus reducing the drift of the temperature process from a to $a\,(k)$ as of $T_L + \Delta T(k, L)$. Although it is theoretically possible to achieve negative emissions by carbon sequestration techniques, we assume that the maximum possible reduction, obtained by starting with a proportion $k = \alpha$ at T_L, is to reduce anthropogenic emissions to zero, as in the pre-industrial state. On the other hand, if $k = 0$, nothing is changed, so $a\,(0) = a$. Equation 5 expresses this relationship.

Thus, the choice of the fraction k affects both the new drift and the delay until the new drift applies. Since a substantial portion of mitigation expenditures takes the form of investments into R&D and technologies for mitigation, we may think of the proportion

$ke^{-\delta(s-T_L)}$ of disposable GDP set aside for mitigation as ensuring that the capital necessary to maintain the temperature drift $a(k)$ is built up and maintained. This may require higher initial efforts, followed but a somewhat reduced capital maintenance effort. This possibility is crudely modelled by parameter δ, as explained earlier.

The associated drift of C_t thus changes at the endogenous time $T_L + \Delta T(k, L)$, from a to $a(k)$. The temperature process' dynamics thus change from Eq. 1 to

$$d\hat{C}_t = a(k)dt + \beta dW_t, \qquad \text{for} \quad t \geq T_L + \Delta T(k, L) \tag{12}$$

To this will correspond a *new* disposable GDP

$$\widehat{DGDP}_t = V_t e^{-\rho(\hat{C}_t - C_P)},, \qquad \text{for} \quad t \geq T_L + \Delta T(k, L) \tag{13}$$

which has the same form as the disposable GDP described in Eq. 3, but which is now a function of a temperature process C_t with a different drift.
Let us note T_L as the first passage time of the temperature at a level L that triggers the decision to use the budget k allocated with the mitigation

$$T_L = \inf\{t \geq 0 : C_t \geq L\} = \inf\{t \geq 0 : Z_t \geq l\}. \tag{14}$$

with $\{Z_t = \gamma t + W_t, t \geq 0\}$ a drifted $\mathbb{P}-$Brownian Motion, $l = \frac{L-C_0}{\beta}$ and $\gamma = \frac{a}{\beta}$. The mitigation budget $V_s e^{-\rho(C_s - C_P)} k e^{-\delta(s-T_L)}$ will be spent as a continuous flow from T_L to the date of the catastrophe, denoted $H^{+}_{L_1,D}$. This is the first time that the temperature process remains without interruption above the temperature level L_1 for D units of time. It's mathematical definition is given in Appendix A.

4 The Objective Function

We are taking the point of view of a global decision maker by attempting to maximize the discounted cumulative future disposable GDP over the next $T = 500$ years by choosing a threshold temperature L that triggers the beginning of the mitigation investment period. At the same time, we want to determine the proportion k of disposable GDP devoted to mitigation at the beginning of the period. The decision to undertake mitigation causes the drift of the temperature process to drop from its historical level a to a lower level $a(k)$ after a delay $\Delta T(k, L)$. The choices of L and k have no effect on the tipping level L_1; however, they affect the date at which L_1 may be reached as well as the probability of a catastrophe, i.e., the probability that temperature stays above L_1 continuously for at least D years. Let us now consider the following cases for L.

1. $\mathbf{L} < \mathbf{L_1}$ - i.e., the endogenous threshold L which triggers the investment in mitigation, is lower than the exogenous threshold L_1 that may trigger catastrophic events. As long as the global social decision maker chooses a mitigation temperature threshold L lower than the catastrophe threshold L_1, and as long as T_L, the first passage time of temperature at a level L, is smaller than the horizon T, he will invest a fraction k of his budget in mitigation at time T_L, causing the temperature process to lower its drift. The decrease in the drift does not happen immediately after the investment is made, but is subject to a time delay equal to $\Delta T(k, L)$. After $T_L + \Delta T(k, L)$[15], and if the temperature process has not yet reached the tipping point level L_1, the temperature process is then both less likely to reach L_1 and less likely to stay above L_1 for a long period of time than in the absence of the mitigation decision. Here, two different things can happen. In fact, it can be that $T_L + \Delta T(k, L)$ is small enough to avoid a catastrophe (Case Ia). Figure 1a illustrates this situation, i.e., the one in which the investment in mitigation has been promptly made. This has caused a decrease in the temperature drift already at early stages, thus avoiding the catastrophe. Conversely, it could happen (Case Ib) that the time $T_L + \Delta T(k, L)$ is not small enough to avoid a catastrophe, as illustrated by Figure 1b.

2. $\mathbf{L} > \mathbf{L_1}$ - i.e., the endogenous threshold L which triggers the investment in mitigation, is higher than the exogenous threshold L_1 that may trigger catastrophic events. If the global decision maker decides to invest at a temperature higher than the level L_1 that triggers the catastrophe, he faces two possible situations. In the first one (Case IIa), it could happen that L, the threshold temperature that triggers an investment k in mitigation, is reached before a possible catastrophe. If this happens, then the global decision maker finds himself in a situation similar to Case I, where mitigation expenditures may still be sufficient to avoid the catastrophe through a timely decrease in temperatures; however, the reduction in drift will need to drive the temperature process below L_1 before D units of time are spent consecutively above the threshold, which is of course less likely than if the threshold had not been reached in the first place, as in Cases Ia and Ib. It can also happen (Case IIb) that D units of time pass without the temperature process touching the mitigation investment threshold L, now higher than the catastrophe threshold L_1. In this case no mitigation procedure is brought forward and the environmental catastrophe occurs in a finite period with higher probability. These two possibilities are illustrated in Figures 1c and 1d.

[15] Note that time $T_L + \Delta T(k, L)$ might be shorter or longer than time T_{L_1}.

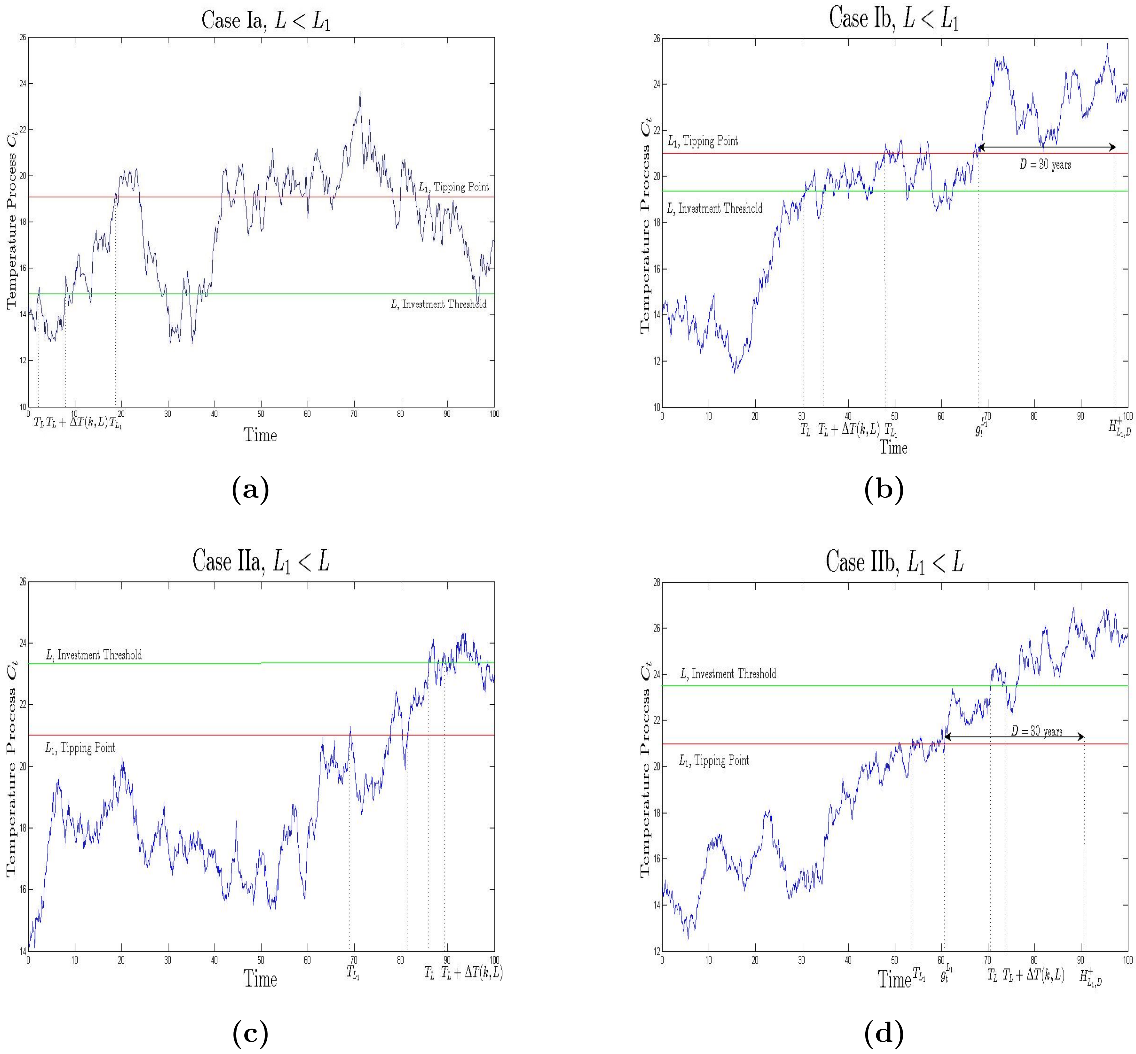


Figure 1: Simulation of one path. The green line represents the level L at which the global social decision maker starts investing in mitigation. The solid red line represents the level L_1, which is the temperature above which the catastrophe can be triggered, if the process stays continuously above L_1 for a period of time at least equal to D.

The four different cases pictured in Figure 1 imply different formulations of the objective function to be maximized by the global decision maker. The global decision maker an optimal investment threshold L^* and an optimal investment fraction k^* such that the expected discounted sum of the future disposable GDP is maximized. In order to do so, he has to find the supremum, over L and k, of a function $f(\cdot,\cdot)$. Because the horizon can be considered infinite, it is known in the options literature that the optimal trigger level L^* is constant (Merton, 1973; Carr et al., 1992; Chesney and Jeanblanc, 2004).

The maximization problem simplifies to

$$\sup_{k,L} f(k,L) \Leftrightarrow \sup_{k,L} \left[\mathbf{1}_{L<L_1} \cdot g_1(k,L) + \mathbf{1}_{L \geq L_1} \cdot g_2(k,L)\right] \tag{15}$$

with:

$$g_1(k,L) = E_{\mathbb{P}}\Bigg[\underbrace{\int_0^{T_L \wedge T} DGDP_u e^{-ru} du}_{I_1 a} + \underbrace{\int_{T_L \wedge T}^{T_L+\Delta T(k,L)\wedge H^+_{L_1,D}\wedge T} DGDP_u \left(1 - ke^{-\delta u}\right) e^{-ru} du}_{I_1 b} + \underbrace{\int_{T_L+\Delta T(k,L)\wedge H^+_{L_1,D}\wedge T}^{H^+_{L_1,D}\wedge T} \widehat{DGDP}_u \left(1 - ke^{-\delta u}\right) e^{-ru} du}_{I_1 c}\Bigg] \tag{16}$$

and

$$g_2(k,L) = E_{\mathbb{P}}\Bigg[\underbrace{\int_0^{T_L \wedge H^+_{L_1,D}\wedge T} DGDP_u e^{-ru} du}_{I_2 a} + \underbrace{\int_{T_L \wedge H^+_{L_1,D}\wedge T}^{T_L+\Delta T(k,L)\wedge H^+_{L_1,D}\wedge T} DGDP_u \left(1 - ke^{-\delta u}\right) e^{-ru} du}_{I_2 b} + \underbrace{\int_{T_L+\Delta T(k,L)\wedge H^+_{L_1,D}\wedge T}^{H^+_{L_1,D}\wedge T} \widehat{DGDP}_u \left(1 - ke^{-\delta u}\right) e^{-ru} du}_{I_2 c}\Bigg] \tag{17}$$

This is a generalized optimal stopping problem with two stochastic processes, one associated with the temperature process and one associated with the GDP process. This maximization process has also a specific feature. The integral bounds are in most cases stopping times of the temperature process C and the functions to be integrated involve the two stochastic processes, C and V. In this real options model, what plays the role of the *strike price* are the integrals preceded by a minus sign in I_1b, I_1c, I_2b, and I_2c. They correspond to mitigation costs. The independent parameters L and k are endogenous. The objective function to be maximized can be intuitively decomposed in the following way:

- $\mathbf{L} < \mathbf{L_1}$, i.e., the endogenous threshold L which triggers the investment in mitigation, is lower than the exogenous *tipping level* L_1 that may trigger catastrophic events.

1a Integral 1a computes the expected discounted sum of future disposable GDPs from the starting point of today until time T_L, when the decision to invest in mitigation is taken by the global social decision maker, or until time $T = 500$ years, whichever comes first.

1b From time T_L until time $T_L + \Delta T(k, L)$, without catastrophe in the meantime, an investment for mitigation has been made, but it is still too early for the drift of the temperature process to shift down. This will happen only at time $T_L + \Delta T(k, L)$ and if the catastrophe has not yet occurred. From time T_L, the expected discounted sum of future disposable GDPs is reduced by an amount equal to the fraction invested in mitigation, here given by k.

1c From time $T_L + \Delta T(k, L)$ until time $H^+_{L_1,D}$, i.e. the moment the catastrophe happens, or until time T, the dynamics of the temperature process have changed, and now have a lower drift. If $H^+_{L_1,D}$ is higher than T, this integral will compute the expected discounted sum of future disposable GDPs until time $T = 500$ years. Conversely, if the temperature process stays above L_1 for at least D years without interruption, before T, then catastrophic effects on the environment will occur, and, consequently, the expected discounted sum of future GDPs will stop at $H^+_{L_1,D}$.

- $\mathbf{L > L_1}$, i.e., the threshold that triggers the investment in mitigation is higher than the exogenous *tipping level* L_1 that may trigger catastrophic events.

2a Integral 2a represents the cumulative GDP from the starting point of today until whichever time happens first: *a*) time T_L, when the decision to invest is taken, *b*) $H^+_{L_1,D}$, when the catastrophic event happens, or *c*) $T = 500$ years. If either *b* or *c* is the case, then all the following integrals are zero, since no investment in mitigation (Integral 2b) or change in temperature drift (Integral 2c) has time to occur and to have an effect on the expected discounted sum of future disposable GDPs, since it is too late.

2b If T_L occurs later than $H^+_{L_1,D}$, integral 2b is zero. Otherwise, this integral corresponds to 1b.

2c Integral 2c computes the expected discounted sum of future disposable GDPs during the period of time that goes from $T_L + \Delta T(k, L)$ to $H^+_{L_1,D}$, which is the time when the catastrophe occurs, or to T. However, Integral 2c is strictly positive only when $H^+_{L_1,D}$ is greater than $T_L + \Delta T(k, L)$.

The function chosen to represent the second part of the objective function allows us to set Integral 2c equal to zero if $H^+_{L_1,D}$ is reached after T_L, but before $T_L + \Delta T(k, L)$. When $H^+_{L_1,D} < T = 500$ is smaller than T_L, this formulation will keep Integral 2b and Integral 2c both with zero value.

5 Calibration and Numerical Results

The starting point of the numerical simulations is the year 2011, and $C_0 = 14.8°C$ $(287.95°K)$ is considered as the baseline temperature. The time horizon chosen is $T = 500$

years[16], with timesteps Δt of one year. Parameter values for the Monte Carlo simulation are given in Table 1 and are discussed below.
The catastrophe threshold L_1 was chosen as $19°C$ that environmental events that could reshape the livability of Earth, such as the interruption of the thermohaline circulation (Bahn et al., 2011; Lenton et al., 2008), can happen with positive probability. These values are in line with IPCC (2013), which estimated a possible temperature increase in the range $1.1°C$ to $6.4°C$. It has also been taken into account that temperature shows strong inertia (see Chen et al., 2011). For this reason, the catastrophe threshold has been rounded up to $5°C$[17] above the pre-industrial temperature C_P. That threshold is also justified by the uncertainty surrounding the volatility of the temperature process. Indeed, Weitzman (2007) argued that the probability of temperatures exceeding the interval estimated by IPCC (2013) is not negligible, so catastrophe is a definite possibility. With respect to the percentage of GDP invested in mitigation and in adaptation, numerical results show that mitigation expenditures should be higher than adaptation expenditures Table 2. This might be far different from what actually happens.[18] The parameters for the Monte Carlo simulation are presented in Table 1.

Table 1: Model Parameters

Parameter	Description	Value	Sensitivity Analysis
C_P	Pre-industrial Global Average Temperature Level	$14.0°C$	-
C_0	2011 Global Average Temperature Level	$14.8°C$	-
k^*	Optimal Mitigation Investment Fraction	-	[0% - 10%]
L^*	Optimal Investment Threshold	-	[14 - 22]
L_1	Catastrophe Threshold	19	-
μ	Initial Drift of the GDP Process	3.0%	-
σ	Volatility of the GDP Process	10%	-
a	Drift of the Temperature Process	$0.035°C$	-
η	Natural Trend of Global Average Temperature	-0.1%	-
α	Parameter Modelling the Impact of Mitigation Efforts on the Temperature Drift	1	-
β	Volatility of the Temperature Process	-	[$0°C$ - $2°C$]
r	Discount Rate	1.5%	[0.0% - 5.0%]
δ	Depreciation Rate of the Mitigation Effort	0	-
ρ	Impact of the Temperature Gap on the Disposable GDP	0.29%	-
θ	Parameter Modelling the Magnitude of the Delay	1	-
Δt	Timesteps of the Processes	1 (year)	-
D	*Parisian* Window	50 (years)	[0; 50]
V_0	GDP in 2011	$69.993 (2011 Trillions)	-
$DGDP_0$	Disposable GDP in 2011	$69.832 (2011 Trillions)	-
T	Horizon	500 years	-

The drift a of the temperature process is positive, given global warming, but very small.

[16]For a detailed explanation of this choice, please refer to Lenton et al. (2008) and IPCC (2013).

[17]While IPCC (2013) considers the interval to be a possible path before the end of the 21^{st} century, here a higher degree of uncertainty has been considered, since scientific results about climate sensitivity and temperature processes have been very heterogeneous.

[18]Indeed, due to the uncertainty involved, countries might prefer to invest in adaptation rather than in mitigation. Said differently, mitigation techniques, at least at the moment, are much slower than adaptation techniques for reaching the desired goals. Governments are also concerned with the problem of *free riding*, which accompanies many mitigation efforts. This point must be stressed. The *local* characterization of many adaptation projects reduces the issue of free riding, and it represents an additional reason why adaptation is preferred.

The rate chosen is $a = 0.035$. Despite the fact that global temperature increased by $0.8°C - 1.0°C$ during the last century, the future is very uncertain given current carbon emissions and polluting trends. The parameter chosen thus corresponds to the lower, conservative, limit of the possible temperature increase in the next 100 years (Lenton et al., 2008), and also to the medium sensitivity as found by Bahn et al. (2012). The volatility of the temperature process was chosen to take different possible values, from $\beta = 0°C$ to $\beta = 2°C$, in order to better reflect the uncertainty that still surrounds climate and temperature models used for forecasting and to better show the influence of such parameter on our results. The outcome of the maximization problem is sensitive to the volatility. Given the variability that surrounds the global average temperature and its anomalies, it did not seem reasonable to limit our analysis to only one fixed standard deviation parameter. Because uncertainty is a core element in climate change, it is necessary to include it in the in the decision maker's optimal choice. Considering that past temperatures and their variability are strongly dependent on frequency, place, and tool of observation (IPCC, 2014; Hansen et al., 2010), a volatility parameter ranging from $\beta = 0°C$ to $\beta = 2°C$ seems reasonable. While standard deviations of $\beta = 0°C$ and $\beta = 2°C$ are unlikely to be observed, it is very useful to test the model's behaviour at the extremes.

The starting GDP value, V_0, was chosen to be \$69.993 trillion US dollars, and the drift μ of the GDP process was set to 3%. Both these values were chosen based on the latest reports on macroeconomic data (Central Intelligence Agency, 2012; The World Bank, 2012), which in particular show an average annual drift in 2011 of 2.2% for developed countries and 4.1% for developing countries. It is likely that more resources are spent on mitigation and adaptation in developed than in developing countries, but this distinction is not taken into account in our analysis. The volatility σ of the GDP process was chosen to be 10% (World Bank Historical GDP Data).
The parameter δ models whether the mitigation effort is a constant proportion of disposable GDP ($\delta = 0$), or if it decreases ($\delta > 0$), or increases ($\delta < 0$) over time. It was chosen to be zero to simplify the optimization process. The parameter D represents the Parisian delay, i.e., the time window during which the temperature process has to stay above the threshold L_1 for the catastrophe to happen.[19] The choice of $D = 50$ years as the time window is somewhat arbitrary. It is a very short time in climatic terms, in line with the notion of a *tipping point* as discussed by Lenton et al. (2008).[20]

[19]The concept of Parisian delay has been borrowed from Parisian options, i.e., financial options whose exercise is triggered by the length of the excursion of a price process above a threshold (Chesney et al., 1997).

[20]While *tipping elements* may be very heterogeneous and not yet entirely known, for the sake of simplicity they are usually considered to trigger the same effects at the same time. Again as in Lenton et al. (2008), only *tipping elements* caused by human activities are taken into consideration.

5.1 Optimal Temperature Threshold and Investment Rate

Given the above parameters, the optimal mitigation investment threshold L^* is determined by allowing candidate values to vary between $14°C$ and $22°C$. Numerical simulations that utilize a grid search methodology compute the expected discounted sum of future global disposable GDPs associated with waiting for temperature to reach a level L^* before investing a fraction k^* in mitigation technologies. The optimal fraction k^* of disposable GDP that Governments can invest in mitigation is chosen by allowing k to vary between 0% and 10%. In fact, higher investment percentages are simply not realistic and the 10% limit never constrains the optimal value.[21] The optimal levels L^* and k^* for various values of β are shown in Table 2. It has been demonstrated empirically that a solution (k^*, L^*) exists and is unique for each set of input parameters.[22]
The optimal mitigation investment threshold varies between a minimum of $14°C$ and a maximum of $15.75°C$ depending on the assumed volatility of the temperature process. This means that the threshold date is already behind us (for low assumed temperature volatilities) or is not far in the future (for higher volatilities). How far in the future? We examine this question below (Table 2).

This result has implications that are manifest in terms other than the timing of the mitigation investment decision. If we look at the optimal fraction of disposable GDP that needs to be invested, year after year, once the decision to mitigate is taken, we find that k^* lies between 1% and 7% depending on the volatility of the temperature process. Furthermore, for small values of the volatility, there is a positive correlation between the optimal temperature threshold and the optimal investment fraction: the higher the temperature threshold, the longer the mitigation decision is postponed (optimally), and the higher the optimal investment effort. This makes intuitive sense since the impact on the temperature drift will need to be stronger if temperature is allowed to come closer to the catastrophe threshold L_1 before any intervention. This positive correlation between the optimal investment threshold L^* and the optimal amount to be invested in mitigation k^* is observed at all reasonable levels of the temperature volatility. It breaks down at unrealistically high volatilities for reasons discussed below.

As implied by the results stated so far, the volatility of the temperature process is a crucial parameter. This is why the optimal levels of L^* and k^* are presented for different possible values of β in Table 2.

[21] Indeed, although we do not model this phenomenon, Bahn et al. (2012) found that highly effective adaptation measures can hinder investments in mitigation in the medium to long term.

[22] In the maximization process, k^* and L^* are jointly and endogenously determined by relying on a grid search method. In case of multiple local maxima for L, we take the supremum, so that our solution is always unique.

Table 2: Simulation Results
$r = 1.5\%$, $D = 50$ years

$\boldsymbol{\beta}$	$\mathbf{L}^*$	$\mathbf{k}^*$
$0°C$	$14.0°C$	1%
$0.1°C$	$14.0°C$	1%
$0.3°C$	$14.75°C$	2.5%
$0.5°C$	$15.0°C$	3%
$0.75°C$	$15.75°C$	4%
$1°C$	$14.9°C$	5%
$1.5°C$	$14.3°C$	6.8%
$2°C$	$14.0°C$	7%

When expressing the optimal investment threshold L^* as a function of the volatility β of the temperature process, it is interesting to note how strong the relationship is between the two. In fact, the level L^* is driven by the uncertainty surrounding future temperature levels in the following way: when uncertainty is fairly low, i.e., $\beta \leq 0.5°C$, it is easier to foresee an increase in temperature levels in the future, given that the drift a of the process is positive, i.e., $a = 0.035$ per unit of time. In this case, it is optimal to invest as soon as possible. Conversely, when the volatility of the temperature process is fairly high, i.e., $\beta \geq 0.50°C$, there is greater uncertainty about future temperature levels. In fact, in this situation, the probability of lower temperatures is higher than before. For $\beta \leq 0.75°C$, results are in line with traditional real options theory: the investment boundary L^* should be an increasing function of the volatility. However, when volatility increases past a critical value, this no longer holds true, representing an interesting result. In this case, the presence of a possible catastrophe induces incentives to mitigate greenhouse gas emissions sooner. Indeed, L^* grows only when β goes from $0°C$ to $0.75°C$. This effect is also different from what has been reported in some of the literature on tipping points (Keller et al., 2004; Brozović and Schlenker, 2011). While we still find a *non-monotonic* relationship between optimal mitigation and uncertainty, as reported in the cited literature, the relationship is reversed in our model. In fact, in the presence of a stochastic temperature process, given the ultimate impact of a catastrophe, i.e., a permanent collapse of the global GDP (see Footnote 8), and a delay between the breach of the tipping point L_1 and the occurrence of the catastrophe, the decision maker faces a trade-off: strong uncertainty makes him cautious when risk increases. In this case, his objective is to invest sooner, since the gain from waiting is not worth the additional expected cost linked to the catastrophe. On the contrary, an increase in the uncertainty level makes the gain from waiting the dominating strategy at lower risk levels.
In the standard real options theory, the optimal boundary is a monotonic function of the risk. However, in some specific cases, in particular for down and out barrier American currency calls, this non-monotonic feature might also be observed. With this specific option, if the value of the currency reaches the barrier, then the option is lost. The loss of the option plays the role of the catastrophe in the framework of this model. With such a barrier option, an increase in volatility generates a higher exercise boundary when

volatility levels are small. However, with higher levels of the volatility, opposite effect appears: the exercise boundary decreases when volatility increases. There is a trade-off between the potential benefit that a volatility increase might generate, i.e., higher profit, and potential risks, i.e., higher probability of losing the option. For small levels of volatility, the first effect dominates; for higher levels, the second one is stronger.
Concerning the effects on the optimal mitigation investment fraction k^*, this leads to increases in β because the more volatile the temperature process is, the stronger the financial effort needs to be in order to bring the temperature process back to acceptable levels. For $\beta > 0.75°C$, the optimal investment fraction k^* keeps increasing but at a slower pace. This behaviour is caused by the lower optimal investment threshold, which allows for smaller increases in k^*.
Figure 2a and Figure 2b illustrate the relationships just mentioned.

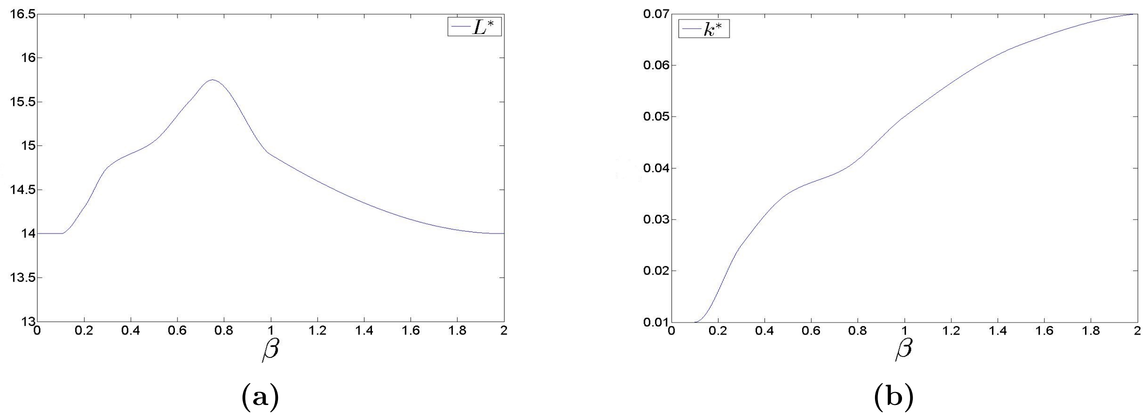
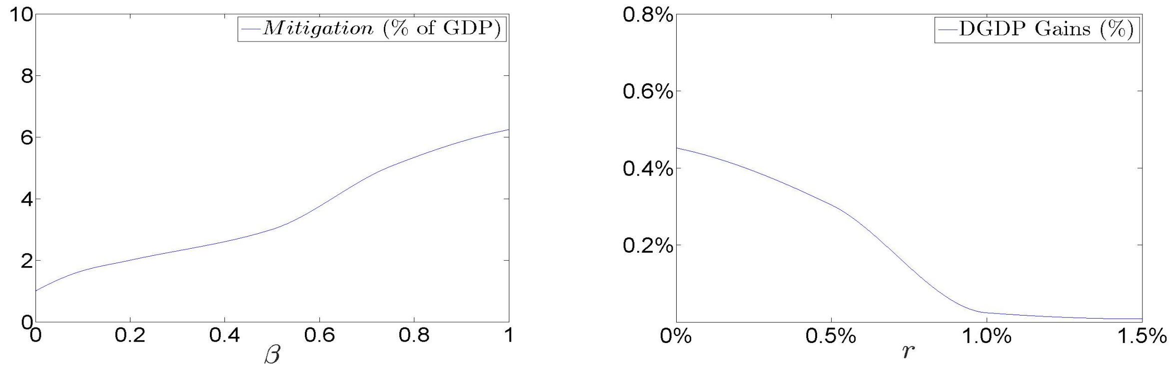


Figure 2: Optimal Investment Threshold L^* (Figure 2a) and Optimal Mitigation Investment Fraction k^* (Figure 2b), plotted against Volatility β, for $D = 50$

The right part of Figure 2a can be justified by looking at the behaviour of the expected catastrophe date as a function of risk, pictured in Figure 6b. When volatility grows, the expected date decreases, making a high -and prompt- monetary investment necessary.

An important relationship to look at is the one occurring at T_L, between the optimal amount to be invested in mitigation k^* and the volatility β. In our model, β is one of the parameters determining the impacts of climate change on the expected discounted sum of future GDP. Optimal mitigation efforts are an increasing function of the temperature volatility. Even with the smallest volatilities, mitigation efforts are at least equal to 1% of the GDP, gross of adaptation, as pictured in Figure 3a. This percentage is much higher than the current one. In many cases, it is also much higher than the exogenous adaptation effort, corresponding to 0.29% in this model. This justifies prompt action in mitigation, in line with the latest international reports on climate change (IPCC, 2014). Figure 3b shows the percentage gains obtained from undertaking mitigation efforts for $\beta = 0.75°C$ in terms of disposable GDP and for different levels of the interest rate r. The graph is obtained by comparing the expected sums of discounted disposable GDP with and without mitigation, i.e., when mitigation efforts correspond to the optimal k^* or to zero. These gains are a decreasing function of the discount rate r. Indeed, when this rate is high, costs generated by a future catastrophe, when discounted, might appear almost negligible. When the discount rate is small, then a strong mitigation effort is required in order to decrease environmental risks. In this case, interests of future generations are taken into account.



(a) Mitigation Efforts, as fraction of GDP, plotted against volatility β

(b) Cumulative DGDP Gains (%) of Mitigation vs. Adaptation only, as function of r and for $\beta = 0.75°C$

Figure 3

The relationship between mitigation and adaptation, and their impacts on the temperature and on the disposable GDP process, is illustrated in Figure 4. The upper part of the plot shows the temperature process and its two different dynamics, without mitigation (green line; $k = 0\%$) and with optimal mitigation (black line; $k^* = 4\%$), for $\beta = 0.75°C$. The lower part of the plot shows the same relationship but expressed in terms of its impacts on the disposable GDP, for $r = 0$. In this case, it is clear that investing in mitigation at time T_L has caused the drift of the temperature process to shift down at time $T_L + \Delta T(k, L)$, from $a(0) = a$ to $a(0.04) < a$. As a consequence, a catastrophe before $T = 500$ years has been avoided. This allows for a higher discounted sum of future GDPs, given by the blue area below the curve. Indeed, despite a lower disposable GDP as of T_L, due to the fact that a fraction $k^* > 0$ has to be invested in mitigation each year, there are positive GDP flows even at later stages. In the particular case pictured, this makes the strategy of investing in mitigation the optimal k^*, i.e., $k^* = 4\%$ for $\beta = 0.75°C$, preferred to the strategy of doing nothing, i.e., of investing in mitigation $k = 0\%$.

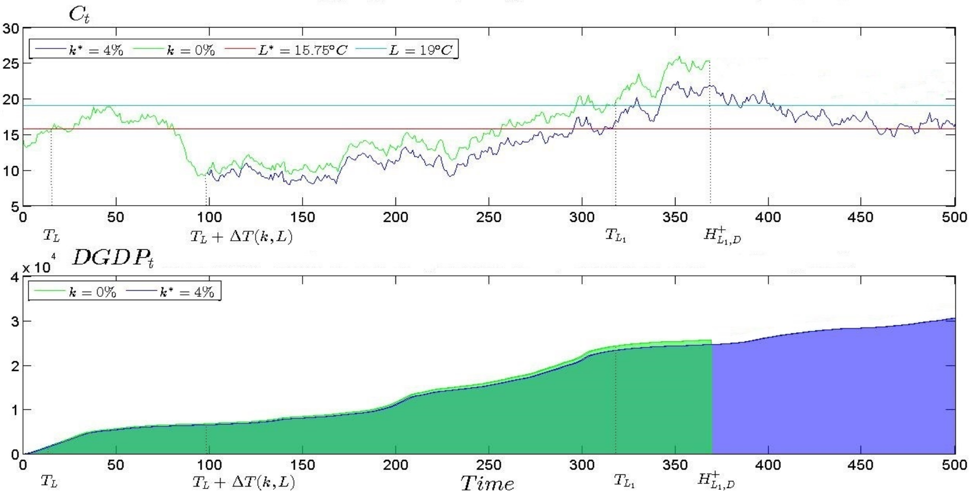


Figure 4: Impacts of Mitigation vs. Adaptation only on the Temperature Process and on the Sum of Discounted Future Disposable Gross Domestic Products, for $\beta = 0.75°C$

As argued in Section 1, the possibility of a climate catastrophe occurring and the impact of mitigation decisions on its probability are arguably the most important questions facing decision makers. For the stochastic processes used in this paper, the catastrophe is certain to happen given a distant enough time horizon. However, it is possible to compute its expected date if nothing is done and its expected date under the optimal policy as well as the sensitivity of these expected dates to parameters. Although the expected date of a catastrophe occurring is reduced by the optimal policy, we find that the possibility of its occurrence within the next 500 years is far from remote for small levels of risk. Figure 7a illustrates this idea. The probability of the catastrophe occurring within the next 500 years is between 100% and 10% depending on β. Given the proximity of that occurrence, we investigate its various determinants below.

Since different values of the exogenous variables can have a substantial impact on the endogenous ones and, consequently, on the simulation results, a sensitivity analysis was conducted. The most relevant results are introduced in the next section.

5.2 Sensitivity Analysis

Monte Carlo simulations were run using different values for the exogenous variables β, D, and r to check their impact on the optimal choice that must be determined by the global decision maker in the presence of global warming. Parameters that need to be considered are the optimal temperature threshold L^* that triggers the optimal investment in mitigation as well as k^*, i.e., the fraction of disposable GDP to be invested in mitigation to achieve the maximum expected discounted sum of future GDPs.

Figure 5 illustrates the maximum values of L^* and k^*, plotted against different values of the Parisian window D, for $\beta = 0.75°C$. It is interesting to notice that, while the optimal threshold L^* remains constant when the excursion length D increases, the optimal fraction of GDP to be invested in mitigation k^* shows a decreasing behaviour. In other words, arguments based on the assumption that long time windows allow for a greater time delay in the social planner's decision-making process do not seem to be justified: a longer window does not imply a greater delay before investing in mitigation, but only a lower fraction k^* of GDP to be devoted to mitigation measures.

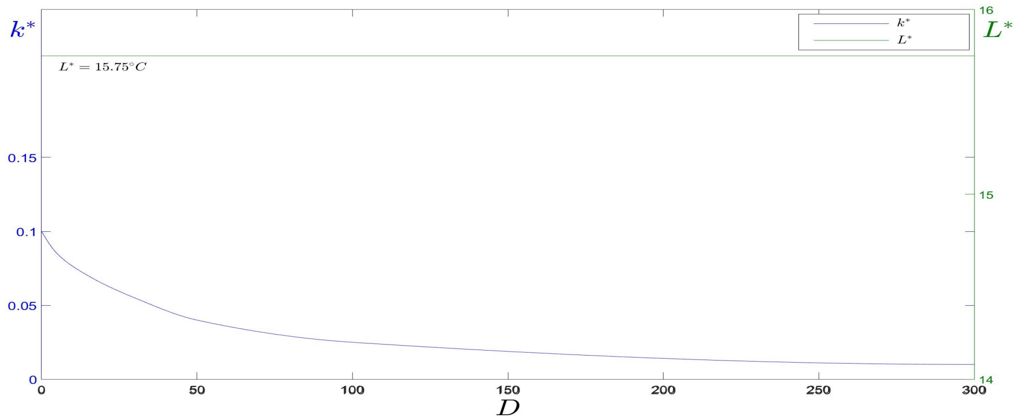
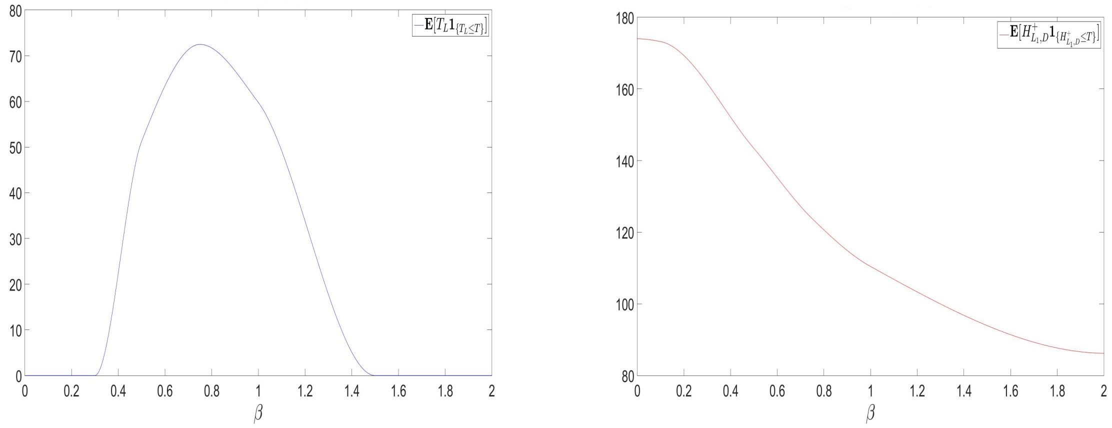


Figure 5: Optimal Mitigation Investment Fraction k^* and Optimal Investment Threshold L^*, plotted against different values of the Parisian Window D, for volatility $\beta = 0.75°C$

As we see in Figure 6a, for small values of the risk, the expected date T_L, i.e. the moment when the threshold L is reached, increases when the volatility β increases. It should not to be forgotten that when $\beta \leq 0.75°C$, the optimal temperature L^*, which triggers the investment in mitigation k^*, increases when β increases. Therefore, it becomes more likely that the barrier L^* is also crossed at later stages. When β is higher than $0.75°C$, L^* starts decreasing again, thus the expected date T_L decreases as well, since crossing the optimal investment threshold might happen sooner. Figure 6b shows the expected date $E\left[H^+_{L_1,D}\right]$ of the environmental catastrophe when it is smaller than T, i.e., when it happens within the chosen time horizon. The expected date decreases when the volatility β increases. Without uncertainty (i.e., $\beta = 0$), this expected date in the business-as-usual scenario is 175 years. As risk increases, a catastrophe might happen sooner and the expectation decreases.[23] This is illustrated in Figure 6b.

[23] As we are not considering trajectories for which $H^+_{L_1,D} > T$.

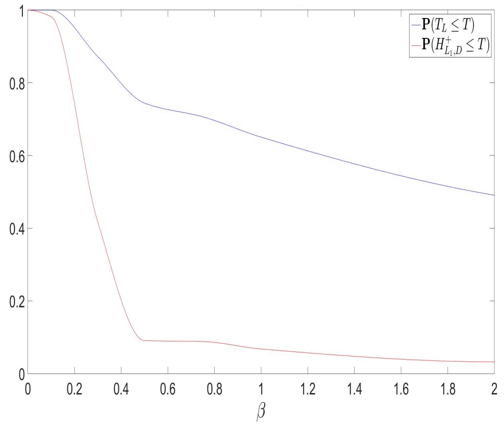


(a) Expected Date T_L plotted against Volatility β, for $D = 50$

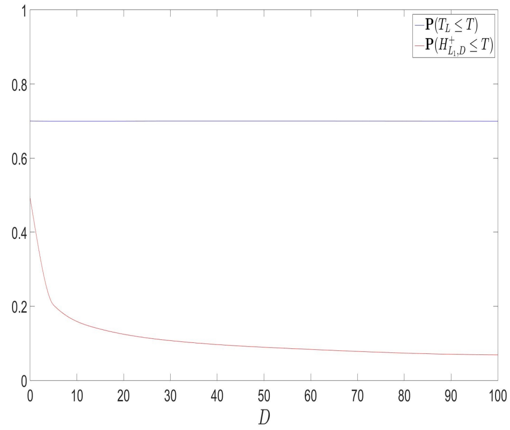
(b) Expected Date of the Environmental Catastrophe $H^+_{L_1,D}$ plotted against Volatility β for $D = 50$

Figure 6

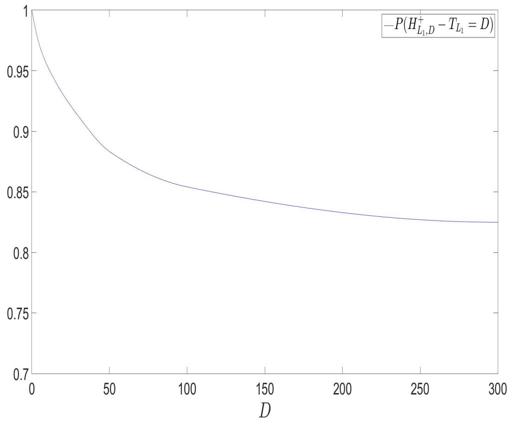
Figure 7a shows the probability that the event T_L, i.e., the moment when the threshold L that triggers the investment in mitigation k, is reached, and that the catastrophic event $H^+_{L_1,D}$ happens within $T = 500$ years, plotted against the volatility β. The probability of both $T_L \le T$ and $H^+_{L_1,D} \le T$ decreases with an increase in the volatility parameter β. This relationship is expected: when the volatility of the temperature process increases, the probability that such a process moves away from the barrier L_1 and a) never touches it or b) decreases and goes back below it increases as well. In addition, as seen in Figure 7a, the probability of the event $H^+_{L_1,D} \le T$ is always smaller than the probability of the event $T_L \le T$, because we set the optimal threshold L^* lower than the tipping point L_1. Figure 7b shows the probability of the event $T_L \le T$, and the probability of the catastrophic event $H^+_{L_1,D} \le T$, plotted against different possible values of the Parisian window D. As expected, the probability of the event $T_L \le T$ remains constant for a given volatility when the Parisian window D increases, since this has no effect on where and when the temperature process crosses the barrier L^* and thus triggers the optimal investment in mitigation k^* from the global decision maker. Conversely, the probability of event $H^+_{L_1,D} \le T$ decreases, going from almost 50% to about 6.9%. Figure 7c illustrates the probability $P(H^+_{L_1,D} - T_{L_1} = D)$, i.e., the probability that a catastrophe will occur D units of time after the first time the temperature process passes the tipping point L_1. As expected, the longer the Parisian window D, the lower the probability $P(H^+_{L_1,D} - T_{L_1} = D)$. However, it can be clearly seen that such a probability remains fairly high, i.e., above 80%, even for very long Parisian windows D. The presence of a large time window does not imply that a catastrophe should be neglected.



(a) Probability of $T_L \leq T$ and Probability of $H^+_{L_1,D} \leq T$, plotted against Volatility β, for $D = 50$



(b) Probability of $T_L \leq T$ and Probability of $H^+_{L_1,D} \leq T$, plotted against different lengths of the Parisian Window D, for $\beta = 0.75°C$



(c) Probability that a Catastrophe occurs D units of Time after T_{L_1}, plotted against different lengths of the Parisian Window D, for $\beta = 0.75°C$

Figure 7

5.3 A Deterministic Temperature Process

Finally, as mentioned in Section 3, various values of α could be used. With 0.1, we obtain the results shown in Figure 8 by relying on the analytical approach explained in Appendix B. It shows that in the deterministic case for the temperature, i.e., when the parameter β is equal to zero, then the drift $a(k)$ becomes negative when the interest rate is small enough, i.e., when interests of future generations are taken into account. In this case, it is optimal to avoid the catastrophe. With a higher discount rate, business as usual in terms of emissions leads to a global catastrophe before the horizon T equals 500 years. Long-term catastrophes are almost negligible today, when discounted at standard levels of interest rates. Unfortunately, only a small discount rate will generate incentives to curb CO_2 emissions and therefore decrease the drift in temperature.

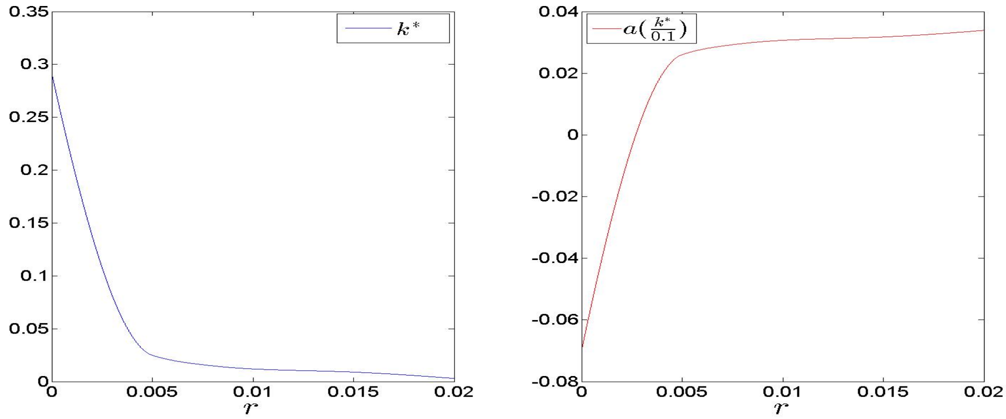


Figure 8: Optimal Mitigation Investment Fraction k^* and Temperature Drift $a(k^*)$, plotted against Interest Rate r, for volatility $\beta = 0°C$ and $\alpha = 0.1$

5.4 Summary of Sensitivities

Sensitivity analyses allow us to shed light on important model results. First of all, as pictured in Figure 5, the length of the Parisian window D does not have a significant impact on the optimal investment threshold L^*. On the contrary, it has an impact on the optimal fraction k^* of GDP to be invested in mitigation, which decreases when the time window increases. This depends on the fact that a higher investment is needed when the Parisian window is small, because the drift of the temperature process needs to be brought down to acceptable levels more quickly. In fact, a higher k^* greatly impacts the drift $a(k)$ of the temperature process.
The expected date $E[T_L]$, which indicates the expectation of the first passage time of the temperature process above the optimal investment threshold L^*, initially increases as volatility β increases. As soon as volatility crosses a critical level, the expected passage time starts decreasing again (Figure 6a). This behaviour closely resembles the behaviour of the optimal temperature threshold L^* when expressed as a function of the volatility β, as can be seen in Figure 2a. The reason can be found in the fact that when the optimal investment threshold L^* decreases, the temperature process might cross it more easily. Conversely, the expected dates $E[T_{L_1}]$ and $E[H^+_{L_1,D}]$, which indicate respectively the expected first passage time of the temperature process above the *tipping point* L_1 and the expected date of the catastrophe, are both monotonically decreasing functions of volatility β. In fact, as volatility β increases, both the event T_{L_1} and $H^+_{L_1,D}$ could happen sooner (see Figure 6b).
The probability of T_{L_1} and $H^+_{L_1,D}$ taking place before $T = 500$ years decrease as volatility increases. In fact, as volatility increases, temperatures are likely to move away from the *tipping point* L_1, either never reaching it, or going back below it once crossed, thus making the occurrence of events T_{L_1} and $H^+_{L_1,D}$ less probable before the horizon T.

6 Conclusion

By applying a real options approach, triggered by the occurrence of a specific event, it has been possible to determine the optimal fraction of disposable GDP and the optimal temperature level at which to invest the fraction so that the expected discounted sum of future GDPs is maximized. For a discount rate of 1.5% and with a volatility of $0.3°C$, Monte Carlo simulations show that governments should invest 2.5% of their disposable GDPs in mitigation when the temperature process hits $14.75°C$ in order for the world to achieve the maximum possible expected discounted sum of future disposable GDPs. Unfortunately, the global temperature level has already reached this threshold. In terms of financial options, the process is already above the optimal investment boundary. If the volatility of the temperature process is equal to $0.75°C$, the optimal investment boundary increases to $15.75°C$. This implies that there is still time left to invest in mitigation and to maximize future financial availability. However, the price to pay is a higher investment fraction that must necessarily be invested in mitigation. The situation where volatility is assumed to be low, i.e., below $0.5°C$, or higher than $0.75°C$, proves to be problematic. In both cases, in fact, the optimal temperature at which the global decision maker should invest k^*% of the disposable GDP is below the current global level, i.e., $C_0 = 14.8°C$. This implies that, depending on the volatility of the temperature process, time is running out or it is already too late to achieve the optimum, and many resources might be destroyed due to global warming. Furthermore, the optimal threshold temperature L^*, at which a fraction k^* of GDP has to be invested in mitigation, is always smaller than the threshold temperature that triggers the catastrophe L_1, justifying prompt action.

Mitigation is always important, and adaptation alone is not sufficient to optimize the expected discounted sum of future GDPs. Mitigation yields higher benefits. While such gains, in terms of a greater expected discounted sum of future GDPs are a decreasing function of the interest rate r, it is easy to notice how this is always achievable when investments in mitigation are undertaken. The optimal mitigation expenditures (2.5% in our standard case) are higher than those for adaptation, i.e., 0.22%. Moreover, investments in mitigation indeed reduce the probability of the occurrence of catastrophic events. The smaller the discount rate r, the higher these investments, and the smaller the probability of such events.

Among the many variables taken into consideration, β, the volatility of the temperature process, is the one that has the most visible impact on the maximization problem and, consequently, on the expected discounted sum of future GDPs. However, at this stage, it is very difficult to assess whether one volatility level is more realistic than another. Uncertainty about the future behaviour of the global temperature process is still high. One thing that many scientists agree on, however, is that even the greatest and strongest effort will not cause a rapid decline in the temperature level. Nevertheless, understanding the true dynamics of the global temperature process, mainly in terms of volatility, has proven to be very important in determining when and how much to invest in mitigation. Failure to undertake the optimal investment leads to suboptimal results, with a great loss of resources.

Nevertheless, further research is needed, mainly for what concerns the specification of a global temperature process and its parameters. The different mitigation and adaptation strategies and possible combinations must be studied to investigate their impacts on the temperature process and on the expected discounted sum of future GDPs. In addition, the positive or negative effects on the GDP of different countries, grouped by income, could be assessed. How one group's mitigation and adaptation choices impact those of other countries could be studied as well, since it is expected that countries with different development patterns will participate differently in GHG reduction efforts.

Appendix A Some Mathematical Tools

For $x = C_0$, we have $C_t = x + at + \beta W_t$, or $C_t = x + \beta Z_t^\gamma$, with Z_t a **drifted** $\mathbb{Q}-$Brownian motion, i.e., $(Z_t = \gamma t + W_t, t \geq 0)$, with $\gamma = \left(\frac{a}{\beta}\right)$.
Let us now define the following functions in terms of T (Chesney et al., 1997) :

$$
\begin{aligned}
T_{L_1}(C) &= \inf\{t \geq 0 : C_t \geq L_1\} && (18)\\
g_t^{L_1}(C) &= \sup\{s \in [0,t] : C_t \geq L_1\} && (19)\\
H^+_{L_1,D}(C) &= \inf\{t \geq 0 : \left(t - g_t^{L_1}(C)\right) \geq D \quad \text{and} \quad C_t \geq L_1\} && (20)
\end{aligned}
$$

or in terms of Z

$$
\begin{aligned}
T_{L_1}(Z) &= \inf\{t \geq 0 : Z_t \geq l_1\} && (21)\\
g_t^{l_1}(Z) &= \sup\{s \in [0,t] : Z_t \geq l_1\} && (22)\\
H^+_{l_1,D}(Z) &= \inf\{t \geq 0 : \left(t - g_t^{l_1}(Z)\right) \geq D \quad \text{and} \quad Z_t \geq l_1\} && (23)
\end{aligned}
$$

with

$$l_1 = \frac{L_1 - C_0}{\beta}.$$

They are, respectively, the first instant the temperature process hits the given level L_1, the last instant before t when this process was at a given level L_1, and the Parisian time, i.e., the first instant when the process spends consecutively D units of time over the level L_1. Notice that $g_t^h(C)$ is not a stopping time. When this random time happens, there is no way to know immediately that it has just happened. We will note $H^+_{L_1,D}$ for $H^+_{L_1,D}(C)$. The mathematical tools useful in this context are the following:

1. The random variables $H^+_{l_1,D}$ and $Z_{H^+_{l_1,D}}$ are independent
2. The law of $Z_{H^+_{l_1,D}}$ is known

$$P(Z_{H^+_{l_1,D}} \in dy) = \frac{dy}{D}\mathbb{I}_{y>L}(y - l_1)\exp\left(-\left(\frac{(y-l_1)^2}{2D}\right)\right) \tag{24}$$

   with $y = (Z_{H^+_{l_1,D}} - Z_{T_{L_1}})$

3. The Laplace transform of $H^+_{L_1,D}$ is given by Chesney et al. (1997)

$$E\left(\exp\left(-\frac{\lambda^2}{2}H^+_{L_1,D}\right)\right) = \frac{\exp(l_1\lambda)}{\Phi(\lambda\sqrt{D})} \tag{25}$$

   where the function Φ is known

$$\Phi(y) = \int_0^{+\infty} z\exp\left(zy - \frac{z^2}{2}\right)dz = 1 + \sqrt{2\pi}ye^{\frac{y^2}{2}}\mathcal{N}(y) \tag{26}$$

   and

$$\mathcal{N}(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{x^2}{2}}dx. \tag{27}$$

Appendix B The Zero Volatility Case

Recall our objective function

$$\sup_{k,L} f(k,L) \Leftrightarrow \sup_{k,L} \left[\mathbf{1}_{L<L_1} \cdot g_1(k,L) + \mathbf{1}_{L\geq L_1} \cdot g_2(k,L)\right] \tag{28}$$

with

$$\begin{aligned} g_1(k,L) = E_{\mathbb{P}} \Bigg[& \underbrace{\int_0^{T_L\wedge T} DGDP_u e^{-ru} du}_{I_1a} + \\ & \underbrace{\int_{T_L\wedge T}^{T_L+\Delta T(k,L)\wedge H^+_{L_1,D}\wedge T} DGDP_u \left(1-ke^{-\delta u}\right) e^{-ru} du}_{I_1b} + \\ & \underbrace{\int_{T_L+\Delta T(k,L)\wedge H^+_{L_1,D}\wedge T}^{H^+_{L_1,D}\wedge T} \widehat{DGDP}_u \left(1-ke^{-\delta u}\right) e^{-ru} du}_{I_1c} \Bigg] \end{aligned} \tag{29}$$

and

$$\begin{aligned} g_2(k,L) = E_{\mathbb{P}} \Bigg[& \underbrace{\int_0^{T_L\wedge H^+_{L_1,D}\wedge T} DGDP_u e^{-ru} du}_{I_2a} + \\ & \underbrace{\int_{T_L\wedge H^+_{L_1,D}\wedge T}^{T_L+\Delta T(k,L)\wedge H^+_{L_1,D}\wedge T} DGDP_u \left(1-ke^{-\delta u}\right) e^{-ru} du}_{I_2b} + \\ & \underbrace{\int_{T_L+\Delta T(k,L)\wedge H^+_{L_1,D}\wedge T}^{H^+_{L_1,D}\wedge T} \widehat{DGDP}_u \left(1-ke^{-\delta u}\right) e^{-ru} du}_{I_2c} \Bigg] \end{aligned} \tag{30}$$

In the absence of volatility, i.e., when $\beta = 0$, and given our set of parameters, as in Table 1, the integrals in Eq. 16 and Eq. Eq. 17 can be solved quasi-analytically.
In order to do so, we will rely on a few facts. First, we know that $min(T_L, T) = T_L$ and that $min(T_L + \Delta T(k,L), T) = T_L + \Delta T(k,L)$. Moreover, since all the integrals are bounded by construction, it is possible to use Fubini's theorem to bring expected values inside the integrals.

Furthermore, we will make use of the fact that, given the dynamics of the temperature process, i.e., $dC_t = adt + \sigma dW_t$, we have $C_t = C_0 + at + \beta W_t$. With respect to the dynamics of the GDP process, i.e., $dV_t = \mu V_t dt + \sigma V_t dB_t$, we obtain $V_t = V_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t}$.

Other useful equalities that will be extensively used are

$$
\begin{aligned}
\delta &= 0 && (31)\\
\alpha &= 0.1 && (32)\\
T_L &= \frac{L - C_0}{a} \leq \frac{22 - 14.8}{0.035} = 205.7 < T = 500 \quad \text{for} \quad L \in [14, 22] && (33)\\
a(k) &= a - (a - \epsilon)\frac{k}{\alpha} && (34)\\
& && (35)
\end{aligned}
$$

B.1 $L < L_1$

- **Integral** $I_1 a$

$$
\begin{aligned}
I_1 a &= E_{\mathbb{P}} \int_0^{T_L \wedge T} DGDP_u e^{-ru} du\\
&= E_{\mathbb{P}} \int_0^{T_L} V_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)u + \sigma B_u} e^{-\rho(C_0 + au - C_P)} e^{-ru} du\\
&= \int_0^{T_L} V_0 e^{(\mu - \rho a - r)u} E_{\mathbb{P}}[e^{-\frac{\sigma^2}{2}u + \sigma B_u}] e^{-\rho(C_0 - C_P)} du
\end{aligned}
$$

We know that $\{e^{-\frac{\sigma^2}{2}t + \sigma B_t}, t \geq 0\}$ is a *martingale*, therefore $E[e^{-\frac{\sigma^2}{2}t + \sigma B_t}] = 1$.
We can then write

$$
\begin{aligned}
I_1 a &= V_0 e^{-\rho(C_0 - C_P)} \int_0^{T_L} e^{(\mu - \rho a - r)u} du\\
&= V_0 e^{-\rho(C_0 - C_P)} \frac{e^{(\mu - \rho a - r)T_L} - 1}{\mu - \rho a - r} && (36)
\end{aligned}
$$

where T_L is given by (33).

- **Integral $I_1 b$**

In Integral 1b, we have to distinguish two different cases, based on whether $T_L + \Delta T(k, L)$ is smaller or greater than $H^+_{L_1,D}$.

$$
\begin{aligned}
I_1 b &= E_{\mathbb{P}} \int_{T_L \wedge T}^{T_L + \Delta T(k,L) \wedge H^+_{L_1,D} \wedge T} DGDP_u \left(1-k\right) e^{-ru} du \\
&= E_{\mathbb{P}} \int_0^{T-T_L} \left(\int_{T_L}^{T_L + t \wedge H^+_{L_1,D}} V_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)u + \sigma B_u} e^{-\rho(C_0 + au - C_P)} \left(1-k\right) e^{-ru} du \right) \mathbb{P}\left(\Delta T(k,L) \in dt\right) \\
&= V_0(1-k) e^{-\rho(C_0 - C_p)} E_{\mathbb{P}} \int_0^{T-T_L} \left(\int_{T_L}^{T_L + t \wedge H^+_{L_1,D}} e^{(\mu - \rho a - r)u} e^{-\frac{\sigma^2}{2}u + \sigma B_u} du \right) \mathbb{P}\left(\Delta T(k,L) \in dt\right)
\end{aligned}
\tag{37}
$$

This integral can be rewritten as

$$
\begin{aligned}
I_1 b = V_0(1-k) e^{-\rho(C_0 - C_p)} \Bigg\{ & E_{\mathbb{P}} \int_0^{\frac{L_1 - L}{a} + D} \left(\int_{T_L}^{T_L + t} e^{(\mu - \rho a - r)u} e^{-\frac{\sigma^2}{2}u + \sigma B_u} du \right) \mathbb{P}\left(\Delta T(k,L) \in dt\right) \\
& + E_{\mathbb{P}} \int_{\frac{L_1 - L}{a} + D}^{T-T_L} \left(\int_{T_L}^{\frac{L_1 - C_0}{a} + D} e^{(\mu - \rho a - r)u} e^{-\frac{\sigma^2}{2}u + \sigma B_u} du \right) \mathbb{P}\left(\Delta T(k,L) \in dt\right) \Bigg\}
\end{aligned}
$$

We can now apply Fubini's theorem to find

$$
\begin{aligned}
I_1 b &= V_0(1-k) e^{-\rho(C_0 - C_p)} \Bigg\{ \int_0^{\frac{L_1 - L}{a} + D} \left(\int_{T_L}^{T_L + t} e^{(\mu - \rho a - r)u} E_{\mathbb{P}} \left[e^{-\frac{\sigma^2}{2}u + \sigma B_u} \right] du \right) \mathbb{P}\left(\Delta T(k,L) \in dt\right) \\
&\quad + \int_{\frac{L_1 - L}{a} + D}^{T-T_L} \left(\int_{T_L}^{\frac{L_1 - C_0}{a} + D} e^{(\mu - \rho a - r)u} E_{\mathbb{P}} \left[e^{-\frac{\sigma^2}{2}u + \sigma B_u} \right] du \right) \mathbb{P}\left(\Delta T(k,L) \in dt\right) \Bigg\} \\
&= V_0(1-k) e^{-\rho(C_0 - C_p)} \Bigg\{ \int_0^{\frac{L_1 - L}{a} + D} \left(\int_{T_L}^{T_L + t} e^{(\mu - \rho a - r)u} du \right) \mathbb{P}\left(\Delta T(k,L) \in dt\right) \\
&\quad + \int_{\frac{L_1 - L}{a} + D}^{T-T_L} \left(\int_{T_L}^{\frac{L_1 - C_0}{a} + D} e^{(\mu - \rho a - r)u} du \right) \mathbb{P}\left(\Delta T(k,L) \in dt\right) \Bigg\}
\end{aligned}
$$

where we relied on the fact that $\left\{ e^{-\frac{\sigma^2}{2}t + \sigma B_t}, t \geq 0 \right\}$ is a *martingale*. We then have

$$
\begin{aligned}
I_1 b &= \frac{V_0(1-k)e^{-\rho(C_0-C_p)}}{\mu-\rho a-r}\Bigg\{\int_0^{\frac{L_1-L}{a}+D}\left(e^{(\mu-\rho a-r)(T_L+t)}-e^{(\mu-\rho a-r)T_L}\right)\mathbb{P}\left(\Delta T(k,L)\in dt\right)\\
&+\int_{\frac{L_1-L}{a}+D}^{T-T_L}\left(e^{(\mu-\rho a-r)(\frac{L_1-C_0}{a}+D)}-e^{(\mu-\rho a-r)T_L}\right)\mathbb{P}\left(\Delta T(k,L)\in dt\right)\Bigg\}\\
&= \frac{V_0(1-k)e^{-\rho(C_0-C_p)}}{\mu-\rho a-r}\Bigg\{\int_0^{\frac{L_1-L}{a}+D}e^{(\mu-\rho a-r)(T_L+t)}\mathbb{P}\left(\Delta T(k,L)\in dt\right)\\
&-e^{(\mu-\rho a-r)T_L}\mathbb{P}\left(\Delta T(k,L)\le\frac{L_1-L}{a}+D\right)\\
&+\left(e^{(\mu-\rho a-r)(\frac{L_1-C_0}{a}+D)}-e^{(\mu-\rho a-r)T_L}\right)\mathbb{P}\left(\frac{L_1-L}{a}+D\le\Delta T(k,L)\le T-T_L\right)\Bigg\}
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbb{P}\left(\Delta T(k,L)\in dt\right) &= \frac{\partial\mathbb{P}\left(\Delta T(k,L)\le t\right)}{\partial t}\\
&= \frac{\partial\mathbb{P}\left(\frac{\theta}{V_{T_L}ke^{-\rho(L-C_P)}}\le t\right)}{\partial t}=\frac{\partial\mathbb{P}\left(V_{T_L}\ge\frac{\theta}{kte^{-\rho(L-C_P)}}\right)}{\partial t}
\end{aligned}
$$

For $K(t)=\frac{\theta}{kte^{-\rho(L-C_P)}}$, we have

$$
\begin{aligned}
\mathbb{P}\left(\Delta T(k,L)\in dt\right) &= \frac{\partial\mathbb{P}\left(V_0e^{\left(\mu-\frac{\sigma^2}{2}\right)T_L+\sigma B_{T_L}}\ge K(t)\right)}{\partial t}\\
&= \frac{\partial\mathbb{P}\left(-\frac{B_{T_L}}{\sqrt{T_L}}\le d_2(t)\right)}{\partial t}=\frac{\partial N(d_2(t))}{\partial t}\\
&= \frac{1}{\sqrt{2\pi}}e^{\frac{-d_2(t)^2}{2}}\cdot\frac{1}{K(t)\sigma\sqrt{T_L}}\cdot\frac{\theta}{kt^2e^{-\rho(L-C_P)}} \qquad (38)
\end{aligned}
$$

and where

$$
d_2(t)=\frac{ln\left(\frac{V_0}{K(t)}\right)+\left(\mu-\frac{\sigma^2}{2}\right)T_L}{\sigma\sqrt{T_L}} \qquad (39)
$$

- **Integral** I_1c

By proceeding along the same lines as Integral I_1b, we have

$$
\begin{aligned}
I_1c &= E_{\mathbb{P}} \int_{T_L+\Delta T(k,L)\wedge H^+_{L_1,D}\wedge T}^{H^+_{L_1,D}\wedge T} \widehat{DGDP}_u \left(1-k\right) e^{-ru} du \\
&= E_{\mathbb{P}} \int_0^{\frac{L_1-L}{a}+D} \int_{T_L+t}^{H^+_{L_1,D}\wedge T} \widehat{DGDP}_u \left(1-k\right) e^{-ru} du \cdot \mathbb{P}\left(\Delta T(k,L) \in dt\right)
\end{aligned}
$$

Indeed, if $T_L + \Delta T(k,L)$ is higher than $\frac{L_1-L}{a} + D$, the integral is equal to zero.

$$
\begin{aligned}
I_1c &= E_{\mathbb{P}} \int_0^{\frac{L_1-L}{a}+D} \int_{T_L+t}^{H^+_{L_1,D}\wedge T} V_0 e^{\left(\mu-\frac{\sigma^2}{2}\right)u+\sigma B_u} e^{-\rho(C_0+a(k)u-C_P)} \left(1-k\right) e^{-ru} du \cdot \mathbb{P}\left(\Delta T(k,L) \in dt\right) \\
&= V_0 \left(1-k\right) e^{-\rho(C_0-C_P)} E_{\mathbb{P}} \int_0^{\frac{L_1-L}{a}+D} \int_{T_L+t}^{H^+_{L_1,D}\wedge T} e^{(\mu-\rho a(k)-r)u} e^{-\frac{\sigma^2}{2}u+\sigma B_u} du \cdot \mathbb{P}\left(\Delta T(k,L) \in dt\right) \\
&= V_0 \left(1-k\right) e^{-\rho(C_0-C_P)} \Bigg\{ E_{\mathbb{P}} \int_0^{\frac{L_1-L}{a}} \int_{T_L+t}^{H^+_{L_1,D}\wedge T} e^{(\mu-\rho a(k)-r)u} e^{-\frac{\sigma^2}{2}u+\sigma B_u} du \cdot \mathbb{P}\left(\Delta T(k,L) \in dt\right) \\
&\quad + E_{\mathbb{P}} \int_{\frac{L_1-L}{a}}^{\frac{L_1-L}{a}+D} \int_{T_L+t}^{H^+_{L_1,D}\wedge T} e^{(\mu-\rho a(k)-r)u} e^{-\frac{\sigma^2}{2}u+\sigma B_u} du \cdot \mathbb{P}\left(\Delta T(k,L) \in dt\right) \Bigg\} \qquad (40)
\end{aligned}
$$

Indeed, when $\Delta T(k,L)$ is smaller than $\frac{L_1-L}{a}$, the temperature process will never reach the tipping point L_1. In this case, the catastrophe is avoided and the upper bound of the second integral is T.
Applying Fubini's theorem and since $\left\{e^{-\frac{\sigma^2}{2}t+\sigma B_t}, t \geq 0\right\}$ is a *martingale*, we have

$$
\begin{aligned}
I_1 c \;=\;& V_0 (1-k)\, e^{-\rho(C_0 - C_P)} \\
& \cdot \Bigg\{ \Bigg[\int_0^{\frac{L_1-L}{a}} \int_{T_L+t}^{T_L+t+\frac{L_1-(L+at)}{a(k)}+D} e^{(\mu-\rho a(k)-r)u} du \cdot \mathbb{P}\left(\Delta T(k,L) \in dt\right) \\
& + \int_{\frac{L_1-L}{a}}^{\frac{L_1-L}{a}+D} \int_{T_L+t}^{T_L+\frac{L_1-L}{a}+D} e^{(\mu-\rho a(k)-r)u} du \cdot \mathbb{P}\left(\Delta T(k,L) \in dt\right) \Bigg] \cdot \mathbf{1}_{a(k)>0} \\
& + \Bigg[\int_0^{\frac{L_1-L}{a}} \int_{T_L+t}^{T} e^{(\mu-\rho a(k)-r)u} du \cdot \mathbb{P}\left(\Delta T(k,L) \in dt\right) \Bigg] \cdot \mathbf{1}_{a(k)\leq 0} \\
& + \Bigg[\int_{\frac{L_1-L}{a}}^{\frac{L_1-L}{a}+D} \int_{T_L+t}^{T_L+\frac{L_1-L}{a}+D} e^{(\mu-\rho a(k)-r)u} du \cdot \mathbb{P}\left(\Delta T(k,L) \in dt\right) \Bigg] \cdot \mathbf{1}_{a(k)\in[X,0]} \\
& + \Bigg[\int_{\frac{L_1-L}{a}}^{\frac{L_1-L}{a}+D} \int_{T_L+t}^{T} e^{(\mu-\rho a(k)-r)u} du \cdot \mathbb{P}\left(\Delta T(k,L) \in dt\right) \Bigg] \cdot \mathbf{1}_{a(k)\in[-\infty,X]} \Bigg\} \qquad (41)
\end{aligned}
$$

Indeed, when $\Delta T(k,L)$ is higher than $\frac{L_1-L}{a}$, the catastrophe can still be avoided, if the time spent by the temperature process above the tipping point L_1 is smaller than D units of time, i.e. if $\Delta T(k,L) - \frac{L_1-L}{a} + \frac{L_1-(L-at)}{a(k)} \leq D$, i.e. if the new drift $a(k)$ is negative and small enough, that is:

$$
a(k) \leq X \qquad (42)
$$

where:

$$
X = \frac{L_1 - L - a\Delta T(k,L)}{D - \Delta T(k,L) + \frac{L_1-L}{a}} \leq 0 \quad \text{for} \quad X \in \left[\frac{L_1-L}{a}, \frac{L_1-L}{a} + D\right] \qquad (43)
$$

When $a(k)$ belongs to $[X,0]$, the temperature drift is not small enough to avoid a catastrophe. The latter will occur at date $T_L + \frac{L_1-L}{a} + D$. If $a(k)$ is smaller than X, then the catastrophe will never occur. In this case, the GDP is maximized until time horizon T.

Then

$$
\begin{aligned}
I_1 c \;=\; & \frac{V_0\,(1-k)\,e^{-\rho(C_0-C_P)}}{\mu-\rho a(k)-r} \\
& \cdot\Bigg\{\Bigg[\int_0^{\frac{L_1-L}{a}} e^{(\mu-\rho a(k)-r)\left(T_L+t+\frac{L_1-(L+at)}{a(k)}+D\right)}\mathbb{P}\left(\Delta T(k,L)\in dt\right) \\
& -\int_0^{\frac{L_1-L}{a}} e^{(\mu-\rho a(k)-r)(T_L+t)}\mathbb{P}\left(\Delta T(k,L)\in dt\right) \\
& +e^{(\mu-\rho a(k)-r)\left(T_L+\frac{L_1-L}{a}+D\right)}\mathbb{P}\left(\frac{L_1-L}{a}\le\Delta T(k,L)\le\frac{L_1-L}{a}+D\right) \\
& -\int_{\frac{L_1-L}{a}}^{\frac{L_1-L}{a}+D} e^{(\mu-\rho a(k)-r)(T_L+t)}\mathbb{P}\left(\Delta T(k,L)\in dt\right)\Bigg]\cdot\mathbf{1}_{a(k)>0} \\
& +\Bigg[e^{(\mu-\rho a(k)-r)T}\mathbb{P}\left(\Delta T(k,L)\le\frac{L_1-L}{a}\right) \\
& -\int_0^{\frac{L_1-L}{a}} e^{(\mu-\rho a(k)-r)(T_L+t)}\mathbb{P}\left(\Delta T(k,L)\in dt\right)\Bigg]\cdot\mathbf{1}_{a(k)\le 0} \\
& +\Bigg[e^{(\mu-\rho a(k)-r)\left(T_L+\frac{L_1-L}{a}+D\right)}\mathbb{P}\left(\frac{L_1-L}{a}\le\Delta T(k,L)\le\frac{L_1-L}{a}+D\right) \\
& -\int_{\frac{L_1-L}{a}}^{\frac{L_1-L}{a}+D} e^{(\mu-\rho a(k)-r)(T_L+t)}\mathbb{P}\left(\Delta T(k,L)\in dt\right)\Bigg]\cdot\mathbf{1}_{a(k)\in[X,0]} \\
& +\Bigg[e^{(\mu-\rho a(k)-r)T}\mathbb{P}\left(\frac{L_1-L}{a}\le\Delta T(k,L)\le\frac{L_1-L}{a}+D\right) \\
& -\int_{\frac{L_1-L}{a}}^{\frac{L_1-L}{a}+D} e^{(\mu-\rho a(k)-r)(T_L+t)}\mathbb{P}\left(\Delta T(k,L)\in dt\right)\Bigg]\cdot\mathbf{1}_{a(k)\in[-\infty,X]}\Bigg\} \qquad (44)
\end{aligned}
$$

B.2 $L \geq L_1$

In order to solve Integrals $I_2 a$, $I_2 b$ and $I_2 c$, all the tools used to solve analytically the previous integrals have been applied.

- **Integral** $I_2 a$

$$
\begin{aligned}
I_2 a &= E_{\mathbb{P}} \int_0^{T_L \wedge H^+_{L_1,D} \wedge T} DGDP_u e^{-ru} du \\
&= V_0 e^{-\rho(C_0 - C_P)} \Bigg\{ E_{\mathbb{P}} \int_0^{T_L} e^{(\mu - \rho a - r)u} e^{-\frac{\sigma^2}{2}u + \sigma B_u} du \cdot \mathbf{1}_{T_L < H^+_{L_1,D}} \\
&\quad + E_{\mathbb{P}} \int_0^{H^+_{L_1,D}} e^{(\mu - \rho a - r)u} e^{-\frac{\sigma^2}{2}u + \sigma B_u} du \cdot \mathbf{1}_{T_L \geq H^+_{L_1,D}} \Bigg\}
\end{aligned}
$$

As opposed to before, here when $T_L \geq H^+_{L_1,D}$, i.e., when $L \geq L_1 + aD$, then $H^+_{L_1,D} = \frac{L_1 - C_0}{a} + D$, which is deterministic.

$$
\begin{aligned}
I_2 a &= V_0 e^{-\rho(C_0 - C_P)} \Bigg\{ \int_0^{T_L} e^{(\mu - \rho a - r)u} du \cdot \mathbf{1}_{L < L_1 + aD} \\
&\quad + \int_0^{H^+_{L_1,D}} e^{(\mu - \rho a - r)u} du \cdot \mathbf{1}_{L \geq L_1 + aD} \Bigg\} \\
&= \frac{V_0 e^{-\rho(C_0 - C_P)}}{\mu - \rho a - r} \\
&\quad \cdot \left[\left(e^{(\mu - \rho a - r)T_L} - 1 \right) \cdot \mathbf{1}_{L < L_1 + aD} + \left(e^{(\mu - \rho a - r)(\frac{L_1 - C_0}{a} + D)} - 1 \right) \cdot \mathbf{1}_{L \geq L_1 + aD} \right] \quad (45)
\end{aligned}
$$

- **Integral** $I_2 b$

If L is high enough, the catastrophe occurs before the temperature level L is reached. In this case, the integral $I_2 b$ is equal to zero, then

$$
I_2 b = I_1 b \cdot \mathbf{1}_{L < L_1 + aD} \quad (46)
$$

- **Integral** $I_2 c$

As in integral $I_2 b$, if L is high enough then the catastrophe occurs before the temperature level L is reached. In this case, the integral $I_2 c$ is equal to zero, so

$$
I_2 c = I_1 c \cdot \mathbf{1}_{L < L_1 + aD} \quad (47)
$$

Appendix C Curbing Emissions and its Effects on Future Temperatures

As can be inferred from Figure 9, it would take a long time after emissions are reduced to acceptable levels before the global mean temperature would reach its pre-industrialization level (Friedlingstein et al., 2011; IPCC, 2013).

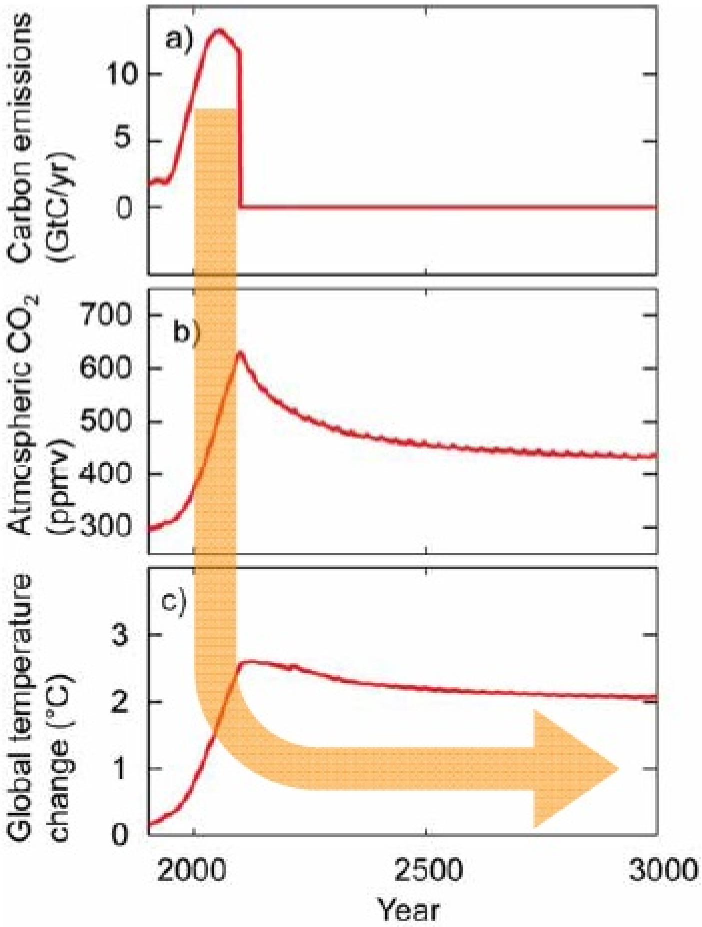


Figure 9: Decay Rate of Temperature as a Function of Emission Reductions

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