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A Parametric Proof for Strategyproofness in the Large of the Probabilistic Serial Mechanism

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Daniel Abächerli
Schwyz, Switzerland

Student-ID: 09-925-009
daniel.abaecherli@gmx.ch

Advisor:

Prof. Sven Seuken, PhD
Department of Informatics
University of Zurich
http://www.ifi.uzh.ch/ce

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Abstract

The *probabilistic serial mechanism (PS)* is one of the most well-understood and studied mechanisms for the assignment problem. While PS is ordinal efficient, it is not strategyproof. However, it has been shown that PS is *partially strategyproof* and that this is the strongest incentive concept satisfied by PS so far. Partial strategyproofness is a parametric incentive concept, where the *degree of strategyproofness* of a mechanism depends on the given setting. Strategyproofness lies at the upper end of the parametric spectrum with a degree of strategyproofness equal to 1.

The main result of this thesis is that the degree of strategyproofness for PS converges to 1 as markets get large. The motivation for this result are calculations of the degree of strategyproofness from (Mennle and Seuken, 2017*c*) and the large market results from (Kojima and Manea, 2010).

This result leads to an elegant, parametric proof that PS is strategyproof in the large. Furthermore, it deepens our understanding of the incentives of PS in large markets and allows us to give upper bounds on the market size, such that a certain degree of strategyproofness for PS can be guaranteed. Similarly, it can also give lower bounds on the degree of strategyproofness for PS, in settings with sufficiently large quotas of objects.

Our secondary result considers settings where there are as many agents as objects and where all objects have unit capacity. In these settings, the set of utility functions which are uniformly relatively bounded indifferent is, in general, not identical to the set of utility functions for which PS is strategyproof. This disproves a conjecture that was proposed in the construction of partial strategyproofness in (Mennle and Seuken, 2017*c*).

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Introduction

The assignment problem concerns itself with the allocation of indivisible objects to self-interested agents. These agents have private preferences over the objects. In the assignment problem monetary transfers are not allowed, therefore, this is a distinct problem compared to auctions or other settings with transferable utility. One common situation where the assignment problem arises is by allocating students to public schools. Here, all students submit their preferences over the different schools. The education department then has to allocate each student to a school (Abdulkadiroğlu and Sönmez, 2003; Abdulkadiroğlu, Pathak and Roth, 2005). Other situations are, for example, allocating entry level positions in labour markets (Roth, 1984; Featherstone, 2011) and allocating accommodation in subsidized housing (Abdulkadiroğlu and Sönmez, 1998).

One of the most well-understood and studied random assignment mechanism is the *probabilistic serial mechanism*, which was first proposed by (Bogomolnaia and Moulin, 2001). As input for this mechanism, the agents report their preference orders over a given set of objects. The output of this mechanism is a probability distribution over this set of objects. This allocation can be computed with the *simultaneous eating algorithm*. For this algorithm, the objects are interpreted as divisible goods of probability shares. Every object has probability shares equal to its quota. The quota or capacity of an object defines to how many agents it can be assigned to. At the start of the algorithm, every agent starts consuming probability shares from its most preferred object at a speed which is identical for all agents. Whenever the quota of an object is exhausted, all agents that were consuming probability shares of this object start consuming probability shares from their next most preferred object that still has a positive capacity. The agents keep eating probability shares at equal speeds based on this rule until they have consumed probability shares that add up to 1 or until no object has any capacity left. The random assignment for each agent is described by the probability shares the agent has consumed from each object. Consider the following example that illustrates this algorithm:

Example 1. *Let there be three objects with unit capacity and three agents that report the following preference orders:*

$$\succ_1 : a \succ b \succ c, \quad \succ_2 : b \succ a \succ c, \quad \succ_3 : b \succ c \succ a. \tag{1.1}$$

The notation $a \succ b$ means that object a is strictly preferred to object b. The eating speeds of all agents are identical and such that at time $t = 1$ each agent has consumed probability *shares that sum up to 1. At the start of the simultaneous eating algorithm (time* $t = 0$*), agents 2 and 3 start consuming probability shares of object* b*, while agent 1 consumes probability shares of object* a*. At* $t = 1/2$ *the object* b *has no more probability shares left, therefore, agent 2 switches to consuming prabability shares from object* a*, which has a remaining capacity of* $1/2$*. Agent 3 starts consuming the untouched object* c*. Then, at time* $t = 3/4$*, all probability shares from object* a *are consumed. Hence, all agents now consume the remaining* $3/4$ *probability shares of object* c *until the capacity of* c *drops to zero. Agent 1 has consumed* $3/4$ *of the probability shares of object* a *and* $1/4$ *of object* c*, hence, it has a chance of* $3/4$ *to receive object* a *and a chance of* $1/4$ *to receive object* c*. We can denote this by* $(3/4, 0, 1/4)$*, which indicates the assignment probability for agent 1 to get objects* a*,* b *and* c *respectively. For agent 2 we get* $(1/4, 1/2, 1/4)$ *and for agent 3 we get* $(0, 1/2, 1/2)$*. Figure 1.1 illustrates the resulting eating schedules.*

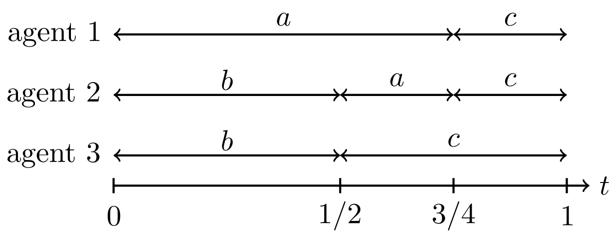


Figure 1.1: Illustration of the eating schedules for PS, given the preferences in (1.1).

From the perspective of a mechanism designer, the probabilistic serial mechanism is interesting, as it satisfies *ordinal efficiency*. The well-known *random serial dictatorship mechanism* only satisfies *ex-post efficiency*, which is a less demanding efficiency concept than ordinal efficiency. Another desired property of mechanisms is *strategyproofness*, i.e., no agent can change the outcome of the mechanism to its advantage by reporting a different preference order than its true one. While the random serial dictatorship mechanism can not be manipulated by such misreports, the probabilistic serial mechanism can be manipulated and is, therefore, not strategyproof. When the probabilistic serial mechanism was introduced by Bogomolnaia and Moulin (2001), they showed that it satisfies *weak stochastic dominance strategyproofness*, which is a less demanding notion of strategyproofness. A mechanism is weakly stochastically dominant strategyproof if the assignment based on truthful reporting is not strictly stochastically dominated by any other assignment resulting from a misreport by one agent.

Mennle and Seuken (2017*c*) recently introduced an intermediate incentive concept called *partial strategyproofness* and showed that it is satisfied by the probabilistic serial mechanism. A mechanism is partial strategyproof, if truthful reporting is a dominant strategy on the subset of all possible von Neumann-Morgenstern utility functions that satisfy *uniformly bounded relative indifference*. A utility function satisfies uniformly bounded relative indifference if, for any two objects, the relative difference in utility is bounded by r. Consider the following examples for uniformly bounded relatively indifferent utility functions.

Example 2. *Let $\hat{O} = \{a, b, c, d\}$ be a set of objects. Let u_i be the following utility function*

$$u_i(a) = 4, \quad u_i(b) = 2, \quad u_i(c) = 1, \quad u_i(d) = 0. \tag{1.2}$$

Then u_i is uniformly bounded relatively indifferent with respect to $r = 1/2$. This is because

$$\frac{u_i(b)}{u_i(a)} = \frac{2}{4} \leq \frac{1}{2}, \quad \frac{u_i(c)}{u_i(b)} \leq \frac{1}{2}, \quad \frac{u_i(d)}{u_i(c)} \leq \frac{1}{2}. \tag{1.3}$$

Analogously, the utility function u'_i where $u'_i(a) = 9, u'_i(b) = 3, u'_i(c) = 1, u'_i(d) = 0$ is uniformly bounded relatively indifferent with respect to $r = 1/3$.

Hence, a mechanism is r-partially strategyproof if truthful reporting is a dominant strategy for the subset of uniformly bounded relatively indifferent utility functions with respect to r. We denote the set of utility functions that satisfy this property by $URBI(r)$. Note that there may exist utility functions that are not uniformly bounded relatively indifferent but for which truthful reporting is a dominant strategy for the given mechanism. This phenomenon is considered in our secondary result.

We can see that if $r = 1$ then $URBI(r)$ is identical to the set of all possible utility functions. Therefore, 1-partial strategyproofness is equal to strategyproofness. At the other end of the spectrum, as r approaches zero, we reach an incentive concept that Mennle and Seuken (2017*c*) called lexicographic dominance strategyproofness (LD-strategyproofness). A mechanism is LD-strategyproof if the assignment based on truthfully reported preferences lexicographically dominates any other assignment with respect to the truthful preference order of the given agent. Therefore, r-partial strategyproofness parametrized the spectrum of incentive concepts between LD-strategyproofness and strategyproofness. The maximal bound r that is satisfied for a mechanism in the given setting is called the *degree of strategyproofness* and is denoted by ρ. Partial strategyproofness is the strongest incentive concept satisfied by the probabilistic serial mechanism so far.

Figure 1.2 illustrates the relation of the different incentive concepts described above.

Strategyproofness

$\Downarrow \; \Uparrow^* r = 1$

r-partial strategyproofness

$\Downarrow \; \not\Uparrow$ | $\not\Uparrow \; \Downarrow^* \rho \to 1$ | $\Downarrow \; \Uparrow^* r \to 0$

Weak SD-SP | SP in the large | LD-SP

Figure 1.2: Relations between incentive concepts (SP = strategyproofness, SD = stochastic dominance, LD = lexicographic dominance).[1]

[1]This figure is adapted from an earlier version of (Mennle and Seuken, 2017*c*).

A different approach is to analyse the incentives of mechanisms in large markets, i.e., when the number of participating agents gets large. A mechanism is *strategyproof in the large* if it is strategyproof in the limit where markets grow large and there are finitely many utility functions the agents can choose from. Strategyproofness in the large was introduced by Azevedo and Budish (2015). They also showed that the probabilistic serial mechanism is strategyproof in the large. Furthermore, a similar result has already been shown by Kojima and Manea (2010). They showed that the probabilistic serial mechanism becomes strategyproof if the quotas of objects are large enough and there are finitely many utility functions the agents can choose from.

Mennle and Seuken (2017*c*) contributed to the analysis of incentives in large markets, by making use of an alternative definition of partial strategyproofness, which enables algorithmic verification. This allowed the authors to calculate the degree of strategyproofness of the probabilistic serial mechanism in different settings. Furthermore, they observed that the degree of strategyproofness ρ may converge to 1 as markets get large. This conjecture is related to, but not implied by the result of Kojima and Manea (2010) and the fact that the mechanism is strategyproof in the large.

The main result of this thesis is that the degree of strategyproofness of the probabilistic serial mechanism converges to 1 as markets get large. More specifically, we show that for any finite setting and any $r \in [0, 1)$, there exists a minimal quota of objects M such that the degree of strategyproofness ρ for the probabilistic serial mechanism is at least as large as r.

This leads to an elegant, parametric proof that the probabilistic serial mechanism is strategyproof in the large. In this parametric proof, we use a result from (Mennle and Seuken, 2017*c*), which states that if the degree of strategyproofness of a mechanism converges to 1 in large settings, then it is strategyproof in the large (This is also indicated in figure 1.2). Moreover, we show that the resulting formula for the minimal quota of objects M can be used to give upper bounds on the minimal quota to ensure r-partial strategyproofness of the probabilistic serial mechanism in a given setting. Conversely, given a setting with large enough quotas of objects, the formula can be used to give a lower bound on the degree of strategyproofness for the probabilistic serial mechanism.

Finally, we considered settings where the number of agents is equal to the number of objects and every object has unit capacity. We found that in these settings, the subset of utility functions for which the probabilistic serial mechanism is strategyproof is, in general, not identical to $URBI(r)$. Thereby, we disprove the conjecture that these sets are identical for the described settings. This conjecture was proposed in the construction of partial strategyproofness in (Mennle and Seuken, 2017*c*).

2

Related Work

Since this thesis builds strongly on the work of Mennle and Seuken (2017*c*) and Kojima and Manea (2010) and others already introduced in section 1, we refrain from mentioning these results here again. Nevertheless, we present other related work on the topic of the assignment problem and the probabilistic serial mechanism.

2.1 The Assignment Problem

Concerning the trade-off of efficiency and strategyproofness, there are two important impossibility results for the assignment problem.

Bogomolnaia and Moulin (2001) showed that for any setting with four or more agents, there can not exist a mechanism that is *strategyproof*, *ordinal efficient* and *treats equals equal*. Equal treatment of equals means that if any two agents report the same preference order then they get identical assignment probabilities for the objects.

Featherstone (2011) showed that *rank efficiency* is incompatible with strategyproofness and even with *weak stochastic dominance strategyproofness*. An assignment is rank efficient if the distribution of ranks across agents can not be stochastically dominated by any other assignment. Note that this is a more demanding concept of efficiency than ordinal efficiency. Aside from this impossibility result, Featherstone (2011) also showed that rank efficient mechanisms can be created using a *linear program*. Finally, he showed that despite the weak incentive concepts that are satisfied by rank efficient mechanism, for some of them truthful reporting is an equilibrium strategy in low information environments.

These results nicely describe the frontier of the current work regarding the assignment problem. We observe that the satisfiable efficiency concepts grow stronger the more we relax the notion of strategyproofness. Therein, we can see the importance of the concept of partial strategyproofness studied in this thesis, as it lies between strategyproofness and weak stochastic dominance strategyproofness.

If we consider mechanisms where agents report their complete von Neumann-Morgenstern utility functions there is another impossibility result from Zhou (1990). No mechanism satisfies strategyproofness, Pareto optimality and symmetry. A mechanism satisfies symmetry if any two agents with identical utility functions have identical utilities for the object assigned to them by the mechanism. Note that in this setting, Pareto optimality is equivalent to ex-ante efficiency (Bogomolnaia and Moulin, 2001).

2.2 The Probabilistic Serial Mechanism

Since Bogomolnaia and Moulin (2001) first introduced the probabilistic serial mechanism (PS), it was extended to the *full preference domain*, i.e., including weak preferences, by Katta and Sethuraman (2006). Moreover, Kojima (2009) extended PS to settings where agents can demand multiple objects.

The probabilistic serial mechanism can be described by using different sets of axioms. Kesten et al. (2011) characterized PS in the general case as the only mechanism which satisfies *ordinal fairness* and *non-wastefulness*. In addition to this, they provided a characterization using *sd-efficiency*, *sd-envy-freeness* and *upper invariance*. Note that "sd" stands for stochastic dominance. Independently, Hashimoto and Hirata (2011) showed that PS can be characterized using sd-efficiency, sd-envy-freeness and *truncation robustness* in the case where the "null-object" always exists. Moreover, they provide a characterization based on the *Rawlsian principle*. In the setting where agents can demand multiple objects, Heo (2014) characterized the extended PS using sd-efficiency, *sd proportional-division lower-bound* and two other auxiliary axioms.

Later, Bogomolnaia and Heo (2012) showed a stronger characterization of PS compared to (Kesten et al., 2011) and (Hashimoto and Hirata, 2011) by using the weaker axiom of *bounded invariance* instead of upper invariance or truncation robustness. Moreover, Bogomolnaia and Heo (2012) made use of the observations in (Heo, 2014) and provided simpler proofs for the results of (Kesten et al., 2011) and (Hashimoto and Hirata, 2011).

Hashimoto et al. (2014) provided the strongest characterization in the general case so far. PS is the only mechanism that satisfies sd-efficiency, sd-envy-freeness and *weak invariance*, where weak invariance is implied by both, bounded invariance and upper invariance. In the case where the null-object always exists, weak invariance is substitute with weak truncation robustness. Weak truncation robustness is implied by truncation robustness and bounded invariance independently.

Finally, Bogomolnaia (2015) characterized PS for the full domain, which even allows for non-integer quotas of objects. Moreover, they showed that in the strict ordinal preference domain, PS is the only mechanism that is sd-efficient, sd-envy-free, and strategyproof on the *lexicographic preference domain extension for lotteries*.

Balbuzanov (2016) showed that PS is *convex strategyproof*, which is implied by partial strategyproofness (Mennle and Seuken, 2017*c*) and implies weak sd-strategyproofness. Che and Kojima (2010) showed that as the number of quotas for each object approach infinity the probabilistic serial mechanism and the random serial dictatorship mechanism become equivalent. This result was generalized by Liu and Pycia (2016), who showed that any ordinal mechanisms that are *asymptotically efficient*, symmetric and *asymptotically strategyproof* lead to identical assignments in large markets.

3

Model

A *random assignment problem* is denoted by $\Gamma = (N, (\succ_i)_{i\in N}, \hat{O}, (q_a)_{a\in\hat{O}})$. The *set of agents* $N = \{1, 2, 3, ...\}$ is indexed by i and finite but arbitrarily large. The *set of proper objects* $\hat{O} = \{a, b, c, ...\}$ is also finite and indexed by j or $a, b, c, \ldots$. Note that there exists a *null object* $\phi \notin \hat{O}$, which is not in $\hat{O}$. By $q = (q_a, q_b, \ldots)$ we denote the *quotas of objects* $j \in \hat{O}$ where $q_j \in \mathbb{N}_{>0}$. The quota of an object is also called its *capacity*. Note that $q_\phi = \infty$.

A *strict preference order* $\succ_i$: $a \succ b \succ c$ denotes the strict preferences of agent i over objects in $O := \hat{O} \cup \{\phi\}$, i.e., $a \succ b$ means that object a is strictly preferred to object b. *Weak preferences* are denoted by $a \succeq_i b$. If the set of objects $\hat{O}$ is fixed, denote by $\mathcal{P}$ the *set of all possible strict preference orders* over $\hat{O}$. A *preference profile* is denoted by $(\succ_i)_{i\in N} = (\succ_1, \succ_2, \succ_3, \ldots)$. When N is fixed we write $\succ$ for $(\succ_i)_{i\in N}$. Denote by $\mathcal{P}^N$ the *set of all possible strict preference profiles* for a given set of agents N.

A *deterministic assignment* is denoted by the matrix $X = (X_{ia})$, where X has $|N|$ rows and $|O|$ columns. For X it holds that $\forall i \in N$, $\forall a \in \hat{O}$, $X_{ia} \in \{0, 1\}$, $\forall i \in N$, $\sum_{a\in O} X_{ia} = 1$ and $\forall a \in O$, $\sum_{i\in N} X_{ia} \leq q_a$. X_{ia} is 1 if agent i receives object a and 0 otherwise. The constraints ensure that every agent only gets one object and every object is assigned at most q_a many times.

A *random assignment* is denoted by a matrix $P = (P_{ia})$, where $\forall i \in N$, $\forall a \in \hat{O}$, $P_{ia} \geq 0$, $\forall i \in N$, $\sum_{a\in O} P_{ia} = 1$ and $\forall a \in O$, $\sum_{i\in N} P_{ia} \leq q_a$. P_{ia} denotes the probability of agent i to receive object a.

Von Neumann-Morgenstern *utility functions* $u_i : O \to [0, 1]$ for deterministic assignments are extended to *expected utility functions* in random assignments by

$$u_i(P) = \sum_{a\in O} u_i(a) \cdot P_{ia}. \tag{3.1}$$

Note that in this thesis we refer to both of them as *utility functions*. A utility function u_i is consistent with $\succ_i$ if $u_i(a) > u_i(b) \Leftrightarrow a \succ_i b$. $U_{\succ_i}$ set of all utility functions u_i consistent with $\succ_i$.

4

PS and Incentive Concepts

In this section we are setting the stage for the formal statement of the main result. This includes the formal definition of the probabilistic serial mechanism (PS) through the symmetric simultaneous eating algorithm, as well as multiple incentive concepts and their relation to PS.

4.1 Probabilistic Serial Mechanism (PS)

The probabilistic serial mechanism (PS) was originally proposed by Bogomolnaia and Moulin (2001). Kojima and Manea (2010) have extended PS to their setting, which is the same as the setting in this thesis. The mechanism is defined through the *symmetric simultaneous eating algorithm*. For this algorithm, each object is viewed as a divisible good of probability shares. At any time $t \in [0, 1]$, every agent "eats" with speed one from its most preferred object among the objects still available. An object a is available as long as the combined consumption of all agents does not exceed the quota q_a of object a. By the time $t = 1$, each agent has consumed a certain amount of probability shares of some objects, this amount corresponds to the PS assignment.

Formally, given a random assignment problem $\Gamma = (N, (\succ_i)_{i\in N}, \hat{O}, (q_a)_{a\in\hat{O}})$, the PS assignment is defined by the symmetric simultaneous eating algorithm as follows: First, define

$$\forall a \in O' \subset O, \quad N(a, O') = \{i \in N \mid a \succeq_i b, \forall b \in O'\}, \tag{4.1}$$

which is the set of all agents whose first preference among the object in O' is a. Set

$$O^0 = O, \quad t^0 = 0, \quad \forall i \in N, \forall a \in \hat{O}, \; P^0_{ia} = 0, \tag{4.2}$$

where O^v denotes the set of available objects at time t^v and P^v_{ia} tracks the consumed probability shares by i of object a up to t^v. Moreover, for all $v \geq 1$, given

$O^0, t^0, (P^0_{ia}), \dots, O^{v-1}, t^{v-1}, (P^{v-1}_{ia})$ define:

$$t^v = \min_{a \in O^{v-1}} \left\{ \max \left\{ t \in [0,1] \mid \sum_{i \in N} P^{v-1}_{ia} + \left| N(a, O^{v-1}) \right| (t - t^{v-1}) \le q_a \right\} \right\}, \tag{4.3}$$

$$O^v = O^v \setminus \left\{ a \in O^{v-1} \mid \sum_{i \in N} P^{v-1}_{ia} + \left| N(a, O^{v-1}) \right| (t - t^{v-1}) = q_a \right\} \Big\}, \tag{4.4}$$

$$P^v_{ia} = \begin{cases} P^{v-1}_{ia} + t^v - t^{v-1} & \text{if } i \in N(a, O^{v-1}) \\ P^{v-1}_{ia} & \text{otherwise.} \end{cases} \tag{4.5}$$

Since O is a finite set, there exists $\bar{v}$ such that $t^{\bar{v}} = 1$. Thus, the assignment of PS is $PS(\succ) := P^{\bar{v}}$. See example 1 in the introduction for a simplified walkthrough of the algorithm.

The crucial step in this algorithm is to identify when the next object will be consumed entirely and, therefore, the agents will have to start consuming another object. This is done in the calculation of t^v. The object that runs out first is determined by evaluating for every object when its capacity will drop to zero, based on the current number of agents consuming it. The smallest of these times will be t^v. The set of the remaining objects is updated accordingly and the random assignment matrix is updated by making use of the fact that the consumption times of probability shares are equal to the assignment probability of an object.

4.2 Strategyproofness and Kojima and Manea (2010)

As mentioned in the introduction, a mechanism is *strategyproof* if no one agent can beneficially manipulate its outcome. The formal definition is the following:

Definition 1 (Strategyproofness). *A mechanism φ is* strategyproof (SP) *if, for all agents $i \in N$, all preference profiles $(\succ_i, \succ_{N \setminus \{i\}}) \in \mathcal{P}^N$, all misreports $\succ'_i \in \mathcal{P}$ and all consistent utility functions $u_i \in U_{\succ_i}$, we have*

$$u_i \left(\varphi \left(\succ_i, \succ_{N \setminus \{i\}} \right) \right) \ge u_i \left(\varphi \left(\succ'_i, \succ_{N \setminus \{i\}} \right) \right). \tag{4.6}$$

Recall that the probabilistic serial mechanism is not strategyproof but satisfies the weaker notion of convex strategyproofness (Balbuzanov, 2016), which implies the even less demanding weak stochastic dominance strategyproofness (Bogomolnaia and Moulin, 2001).

Kojima and Manea (2010) investigated the incentives of PS in large markets. They found that for any given agent, PS is strategyproof if there are sufficiently many copies of each object, i.e., the quotas of the objects are high enough.

Fact 1 (Theorem 1, (Kojima and Manea, 2010)). *Let u_i be an expected utility function consistent with a preference $\succ_i$.*

(i) There exists M such that if $q_a \geq M$ for all $a \in \hat{O}$, then

$$u_i\left(PS\left(\succ_i, \succ_{N\setminus\{i\}}\right)\right) \geq u_i\left(PS\left(\succ'_i, \succ_{N\setminus\{i\}}\right)\right) \tag{4.7}$$

for any preference $\succ'_i \in \mathcal{P}$, any set of agents $N \ni i$ and any preference profile $\succ_{N\setminus\{i\}} \in \mathcal{P}^{N\setminus\{i\}}$.

(ii) Claim (i) is satisfied for $M = xD/d$, where $x \approx 1.76322$ solves $xln(x) = 1$, $D = \max_{a\succeq_i b\succeq_i \phi} u_i(a) - u_i(b)$, and $d = \min_{a\succ_i b, a\succeq_i \phi} u_i(a) - u_i(b)$.

This means that for a fixed utility function and if the quotas of all objects are large enough, PS is strategyproof for any agent with this utility function. However, PS is not necessarily strategyproof for other agents with other utility functions in the same setting. This argument can be extended to any finite number of utility functions.

Fact 2 (Corollary, (Kojima and Manea, 2010)). *Suppose that the set $\hat{O}$ of proper object types and the set $\mathcal{U}$ of expected utility functions on the outcome of PS over the given objects O are fixed and finite. There exists M such that if $q_a \geq M$ for all $a \in \hat{O}$, then for any set of participating agents, truth-telling is a weakly dominant strategy in the probabilistic serial mechanism for every agent whose utility function is in $\mathcal{U}$.*

From Fact 2, Kojima and Manea (2010) deduced that PS becomes strategyproof in large assignment problems, where the utility functions of all agents belong to a finite set.

4.3 Strategyproofness in the Large

Azevedo and Budish (2015) considered the incentives of mechanisms in large settings in general and formalized the intuition that the manipulative power of a single agent diminishes as there is more and more competition for the objects. They called this concept *strategyproofness in the large*. Note that this is a generalization of the insight Kojima and Manea (2010) deduced from Fact 2 for PS.

In contrast to the settings above, Azevedo and Budish (2015) takes an *interim perspective* on the assignment problem. This means the agents do not have full knowledge of the reports of other agents. This uncertainty is addressed by using probability distributions over all possible reports for other agents. The original definition of strategyproofness in the large requires that truthful reporting is the *best strategy in expectation* against any probability distribution of other agents' reports. The reports of other agents are drawn identically and independently from the same distribution.

Note that strategyproofness in the large is a more demanding incentive concept than *approximate Bayes-Nash incentive compatibility.* Strategyproofness in the large demands a best strategy for any probability distribution of reports, while for approximate Bayes-Nash incentive compatibility, truthful reporting is only the best strategy in the probability distribution associated with the Bayes-Nash equilibrium setting (Azevedo and Budish, 2015).

We will adapt the definition of strategyproofness in the large to our setting with an *ex-post* perspective. Hence, we remove the uncertainty of other agents' reports and assume complete knowledge of their reports. From this perspective, the condition of strategyproofness in the large has to be satisfied for all possible reports of other agents, rather than just in expectation for reports drawn i.i.d. from a common distribution. Therefore, the ex-post version of strategyproofness in the large is more demanding than the original interim definition by Azevedo and Budish (2015).

Before we formally define strategyproofness in the large, we need to define how exactly markets get large. Settings get large by increasing the number of agents. However, there are additional conditions that have to hold to make the resulting setting an assignment problem which is consistent with our model.

We adopt the notion of large settings from Kojima and Manea (2010). Fix a set of objects. If the number of agents increases, the number of objects has to increase as well such that at least one object can be assigned to any agent. This means that the sum of all quotas has to be larger or equal to the number of agents in the assignment problem. While this would be achievable if we just keep adding copies of one specific object, we want the quotas to grow more uniformly. Therefore, as the number of agents gets larger, not only the sum of the quotas has to get larger but also the minimum of the quotas.

Formally, consider a sequence of settings $(N^n, \hat{O}^n, q^n)_{n\geq 1}$, where $\hat{O}^n = \hat{O}$ and $n = |N^n|$. We require the following two conditions to hold:

$$\min_{j\in\hat{O}} q_j^n \xrightarrow{n\to\infty} \infty \quad \text{and for any } n, \quad \sum_{j\in\hat{O}} q_j^n \geq n. \tag{4.8}$$

A mechanism is strategyproof in the large, if the mechanism is strategyproof in the limit where the number of agents approaches infinity ($n \to \infty$) and the agents choose their utility functions from a finite set. Formaly,

Definition 2 (SP in the Large, adapted from (Azevedo and Budish, 2015))**.** *Given a fixed, finite set of utility functions $\{u^1, \dots, u^K\}$ and a sequence of settings $(N^n, \hat{O}^n, q^n)_{n\geq 1}$ which get large as specified above. Then, a mechanism φ is* strategyproof in the large (SP-L) *if, for any $\varepsilon > 0$, there exits $n_0 \in \mathbb{N}$ such that for all settings where $n \geq n_0$, no agent with a utility function from $\{u^1, \dots, u^K\}$ can gain more than ε by misreporting.*

Note that PS is strategyproof in the large, as this definition of SP-L is consistent with the reasoning of Kojima and Manea (2010) that PS is strategyproof in large settings.

4.4 Partial Strategyproofness

Mennle and Seuken (2017*c*) introduced the strongest incentive concept satisfied by PS so far, which is *partial strategyproofness*. In this section, we present the results from (Mennle and Seuken, 2017*c*), which are necessary for stating and discussing of our main result.

After the formal definition of partial strategyproofness, we present the decomposition of partial strategyproofness. Then, we define the degree of strategyproofness and present how it can be calculated for small settings, using an alternative definition of partial strategyproofness.

4.4.1 Definition

In order to define partial strategyproofness we need the concept of *uniformly relatively bounded indifferent* utility functions.

Definition 3 (*URBI*(r), from (Mennle and Seuken, 2017*c*))**.** *A utility function* u_i *satisfies* uniformly relatively bounded indifference with respect to bound $r \in [0,1]$ *(short: URBI(r)) if, for all objects* $a, b \in \hat{O}$ *with* $u_i(a) > u_i(b)$*, we have*

$$r \cdot \left(u_i(a) - \min_{j \in \hat{O}} u_i(j) \right) \geq u_i(b) - \min_{j \in \hat{O}} u_i(j) \tag{4.9}$$

We write $u_i \in URBI(r)$ if u_i satisfies uniformly relatively bounded indifference with respect to bound r.

Informally, a utility function is uniformly relatively bounded indifferent with respect to $r \in [0,1]$ if the relative difference in utility of any two consecutive objects is bounded by r.

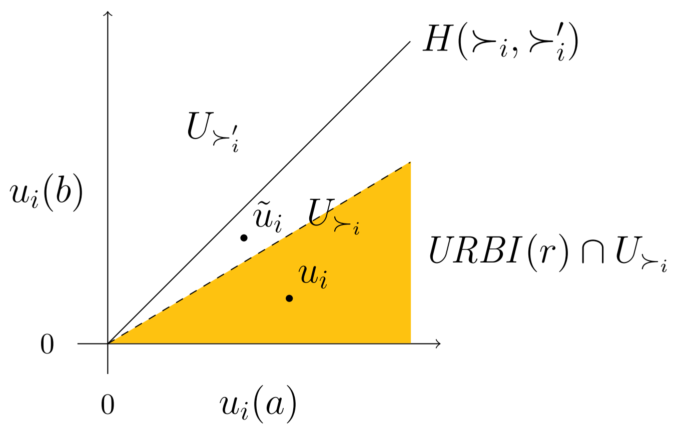


Figure 4.1: Geometric interpretation of *URBI*(r), adapted from (Mennle and Seuken, 2017*c*).

Figure 4.1 provides a geometric interpretation of $URBI(r)$. Consider the preference order $\succ_i$: $a \succ b$, then the set of consistent utility functions $U_{\succ_i}$ is the area under the indifference hyperplane $H(\succ_i, \succ_i')$. If a utility function satisfies $URBI(r)$ it has to be r-relatively uniformly bounded away from the indifference hyperplane. This is true for all utility functions in the shaded area, e.g., for the utility function labeled u_i. Any utility function that is consistent with $\succ_i$ but does not lie in the shaded area, e.g., the point labeled $\tilde{u_i}$, does not satisfy $URBI(r)$. Note that the dashed line which binds $URBI(r)$ away from the indifference hyperplane has slope r. Therefore, we can see that if $r = 1$ we get $URBI(r) = U_{\succ_i}$. See example 2 in the introduction for examples of specific utility functions that satisfy uniformly relatively bounded indifference.

A mechanism is partially strategyproof if it is strategyproof for all uniformly relatively bounded indifferent utility functions.

Definition 4 (Partial Strategyproofness, from (Mennle and Seuken, 2017*c*))**.** *Given a setting* $(N, \hat{O}, q)$ *and a bound* $r \in [0, 1]$*, a mechanism* φ *is* r-partially strategyproof (in the setting $(N, \hat{O}, q)$) *if, for all agents* $i \in N$*, all preference profiles* $(\succ_i, \succ_{N\setminus\{i\}}) \in \mathcal{P}^N$*, all misreports* $\succ_i' \in \mathcal{P}$ *and all utility functions* $u_i \in U_{\succ_i} \cap URBI(r)$*, we have*

$$u_i\left(\varphi\left(\succ_i, \succ_{N\setminus\{i\}}\right)\right) \geq u_i\left(\varphi\left(\succ_i', \succ_{N\setminus\{i\}}\right)\right). \tag{4.10}$$

Recall from section 1 that 1-partial strategyproofness is identical to strategyproofness. Moreover, as r approaches zero, we reach lexicographic dominance strategyproofness (LD-strategyproofness). Therefore, r-partial strategyproofness parametrized the spectrum of incentive concepts between LD-strategyproofness and strategyproofness. Figure 1.2 in section 1 illustrates the relation of partial strategyproofness to some other incentive concepts.

4.4.2 Decomposition

Next, we will take a look at the decomposition of Partial strategyproofness. This decomposition will us allow to differentiate partial strategyproofness from regular strategyproofness and makes it easy to see that PS is partial strategyproof.

In order to define the building blocks of the decomposition, we introduce some notions from (Mennle and Seuken, 2017*c*).

The *neighbourhood of* $\succ_i$, denoted by $\mathcal{N}_{\succ_i} \subset \mathcal{P}^N$, is the set of strict preference orders that differ from $\succ_i$ by only one swap of consecutively ranked objects.

Moreover, we define $U(a, \succ_i) = \{b \in \hat{O} \mid b \succ_i a\}$ as the *upper contour set* of object $a \in O$ at $\succ_i$. This is the set of objects which are strictly preferred to a under $\succ_i$. Analogously, $L(a, \succ_i) = \{b \in \hat{O} \mid a \succ_i b\}$ is the *lower contour set* of object $a \in O$ at $\succ_i$.

Swap monotonicity restricts how the assignment probabilities of a mechanism may change if two consecutive objects are swapped.

Definition 5 (Swap Monotonicity, from (Mennle and Seuken, 2017*c*))**.** *A mechanism φ is* swap monotonic *if, for all agents $i \in N$, all preference profiles $(\succ_i, \succ_{N\backslash\{i\}}) \in \mathcal{P}^N$, all misreports $\succ_i' \in \mathcal{N}_{\succ_i}$ in the neighbourhood of $\succ_i$ such that $a \succ_i b$ but $b \succ_i' a$, one of the following holds:*

- *either:* $\varphi_i\left(\succ_i, \succ_{N\backslash\{i\}}\right) = \varphi_i\left(\succ_i', \succ_{N\backslash\{i\}}\right)$,
- *or:* $\varphi_{ia}\left(\succ_i, \succ_{N\backslash\{i\}}\right) > \varphi_{ia}\left(\succ_i', \succ_{N\backslash\{i\}}\right)$ *and* $\varphi_{ib}\left(\succ_i, \succ_{N\backslash\{i\}}\right) < \varphi_{ib}\left(\succ_i', \succ_{N\backslash\{i\}}\right)$.

If a mechanism is swap monotonic and one considers misreporting by swapping two consecutive objects in ones preference order, then the assignment probability of these objects will either stay the same or the probability of the object that was swapped upwards will strictly increase, while the probability of the other object will strictly decrease.

While still considering consecutive swaps for misreporting, *upper* and *lower invariance* define a behaviour of the assignment probabilities for objects that are strictly less or strictly more preferred than the swapped objects.

Definition 6 (Upper Invariance, from (Mennle and Seuken, 2017*c*))**.** *A mechanism φ is* upper invariant *if, for all agents $i \in N$, all preference profiles $(\succ_i, \succ_{N\backslash\{i\}}) \in \mathcal{P}^N$, all misreports $\succ_i' \in \mathcal{N}_{\succ_i}$ with $a \succ_i b$ but $b \succ_i' a$, we have that $\varphi_{ib}\left(\succ_i, \succ_{N\backslash\{i\}}\right) = \varphi_{ib}\left(\succ_i', \succ_{N\backslash\{i\}}\right)$ for all $b \in U(a, \hat{O})$.*

If a mechanism is upper invariant, then swapping two consecutive objects in the preference order will not affect the assignment probabilities of the objects ranked above these two objects. Analogously, if the mechanism is lower invariant, such a swap will not affect the assignment probabilities of objects ranked below the swapped objects.

Definition 7 (Lower Invariance, from (Mennle and Seuken, 2017*c*))**.** *A mechanism φ is* upper invariant *if, for all agents $i \in N$, all preference profiles $(\succ_i, \succ_{N\backslash\{i\}}) \in \mathcal{P}^N$, all misreports $\succ_i' \in \mathcal{N}_{\succ_i}$ with $a \succ_i b$ but $b \succ_i' a$, we have that $\varphi_{ib}\left(\succ_i, \succ_{N\backslash\{i\}}\right) = \varphi_{ib}\left(\succ_i', \succ_{N\backslash\{i\}}\right)$ for all $b \in L(a, \hat{O})$.*

Mennle and Seuken (2017*c*) showed that these three concepts are a decomposition of strategyproofness.

Fact 3 (Decomposition of SP, from (Mennle and Seuken, 2017*c*))**.** *A mechanism φ is strategyproof if and only if it is swap monotonic, upper invariant and lower invariant.*

Furthermore, partial strategyproofness can be decomposed into swap monotonicity and upper invariance.

Fact 4 (Decomposition of Partial Strategyproofness, from (Mennle and Seuken, 2017*c*))**.** *Given a setting $(N, \hat{O}, q)$, a mechanism φ is* partially strategyproof (i.e., r-partially strategyproof for some $r > 0$) *if and only if it is swap monotonic and upper invariant.*

This shows that partial strategyproofness is a weaker concept than stragetyproofness. Moreover, one can easily verify that PS is swap monotonic and upper invariant. Hence, PS is partially strategyproof.

Fact 5 (Proposition 3, (Mennle and Seuken, 2017*c*)). *Given a setting* $(N, \hat{O}, q)$*, the probabilistic serial mechanism is* r-partially strategyproof *for some* $r > 0$.

4.4.3 Degree of Strategyproofness ($\rho_{(N,\hat{O},q)}$)

If a mechanism is r-partially strategyproof, then by definition it is also r'-partially strategyproof, where $0 < r' < r \leq 1$. Naturally, we are interested in the largest bound r for which a mechanism is partially strategyproof. This largest bound is well-defined and is called the *degree of strategyproofness*.

Definition 8 (Degree of Strategyproofness, from (Mennle and Seuken, 2017*c*)). *Given a setting* $(N, \hat{O}, q)$ *and a mechanism* φ *the* degree of strategyproofness (in the setting $(N, \hat{O}, q)$ *is defined as*

$$\rho_{(N,\hat{O},q)}(\varphi) = \max\{r \in [0,1] \mid \varphi \textit{ is } r\textit{-partially strategyproof in } (N, \hat{O}, q)\}. \tag{4.11}$$

To see that this maximum is well-defined, note that $URBI(r)$ is topologically closed. Therefore, if a mechanism is r'-partially strategyproof for all $r' < r$, then it must also be r-partially strategyproof (Mennle and Seuken, 2017*c*).

Furthermore, Mennle and Seuken (2017*b*) provided an additional definition of partial strategyproofness which allows for algorithmic verification of partial strategyproofness and calculating the degree of strategyproofness for small markets. This additional definition uses a generalized version of stochastic dominance, which is called *r-dicscounted dominance*.

Definition 9 (r-Discountd Dominance (r-DD), from (Mennle and Seuken, 2017*b*)). *For a bound* $r \in [0,1]$ *and a preference order* $\succ_i \in \mathcal{P}$*, where* $\succ_i$: $j_1 \succ j_2 \succ \cdots \succ j_{|\hat{O}|}$ *and assignment vectors* x_i, y_i *we say that* x_i r-discounted dominates y_i at $\succ_i$ *if for all ranks* $K \in \{1, \ldots, |\hat{O}|\}$*, we have*

$$\sum_{k=1}^{K} x_{i,j_k} \geq \sum_{k=1}^{K} y_{i,j_k}. \tag{4.12}$$

Note that for $r = 1$ this is equivalent to stochastic dominance. For any $r < 1$ the assignment probabilities of the object in rank k of the preference order $\succ_i$ is discounted by the k^{th} power of r.

Recall, that stochastic dominance is used to define SD-strategyproofness which is equivalent to strategyproofness. Analogously, r-DD can be used to define r-DD strategyproofness, which is equivalent to partial strategyproofness as Mennle and Seuken (2017*b*) have shown.

Definition 10 (r-DD-Strategyproofness, adopted from (Mennle and Seuken, 2017*b*))**.** *Given a setting* $(N, \hat{O}, q)$ *and a bound* $r \in [0, 1)$*, a mechanism* φ *is* r-DD-strategyproof *if, for all agents* $i \in N$*, all preference profiles* $(\succ_i, \succ_{N\setminus\{i\}}) \in \mathcal{P}^N$ *and all misreports* $\succ_i' \in \mathcal{P}$*,* $\varphi(\succ_i, \succ_{N\setminus\{i\}})$ *r-discount dominates* $\varphi(\succ_i', \succ_{N\setminus\{i\}})$ *at* $\succ_i$*.*

Fact 6 (Proposition 7, (Mennle and Seuken, 2017*b*))**.** *Given a setting* $(N, \hat{O}, q)$ *and a bound* $r \in [0, 1)$*, a mechanism* φ *is r-partially strategyproof if and only if it is r-DD-strategyproof.*

In the first definition of partial strategyproofness it would be necessary to check inequalities for an infinite set of utility functions in order to verify if a mechanism is r-partially strategyproof, which can not be achieved by any implementation. However, since there are finitely many preference profiles and misreports in any finite setting, there are finitely many possible assignments. Therefore, a procedure can be implemented to test the inequalities of r-DD strategyproofness in small markets to verify if a mechanism is partially strategyproof. The markets need to be small since the procedure has to iterate through all possible preference profiles and the respective misreports. The number of misreports grows exponentially in the number of objects. The number of preference profiles grows exponentially in the number of agents and the number of objects. Therefore, calculations for large markets are infeasible. Moreover, the inequalities that define r-DD can be used to calculate the degree of strategyproofness for a given mechanism.

5

Main Result

In this section, the main result of this thesis is presented, which is that the degree of strategyproofness of PS, $\rho_{(N,\hat{O},q)}(PS)$, converges to 1 as the settings get large. We first consider related work to learn if this result was expected, before presenting the result itself. Then, we show how it leads to a parametric proof of the fact that PS is strategyproof in the large, before discussing other implications.

5.1 Convergence of $\rho_{(N,\hat{O},q)}(PS)$

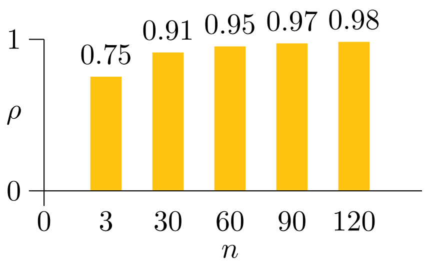
Recall that markets get large in a uniform way with respect to the quotas of objects. This means that in a sequence of settings $(N^n, \hat{O}^n, q^n)_{n\geq 1}$, where $\hat{O}^n = \hat{O}$ and $n = |N^n|$, we not only require that the demand of the agents is met ($\sum_{j\in\hat{O}} q_j^n \geq n$), but also that the minimal quota of objects approaches infinity as n approaches infinity ($\min_{j\in\hat{O}} q_j^n \xrightarrow{n\to\infty} \infty$).

Kojima and Manea (2010) as well as Azevedo and Budish (2015) considered the incentive properties of PS in large markets. (Azevedo and Budish, 2015) showed that PS is strategyproof in the large, i.e., for a finite set of utility functions from which the agents can choose, in the limit where $n \to \infty$, PS is strategyproof. Kojima and Manea (2010) showed that if there are enough copies of each object and only finitely many utility functions to choose from, then PS is strategyproof in large settings.

Since PS is strategyproof for large settings and a finite set of utility functions, it can be expected that this behaviour may also be realized for an infinitely large set of utility functions.

Mennle and Seuken (2017*c*) introduced the incentive measure of partial strategyproofness, which comes with a bound ρ called the degree of strategyproofness. The bound ρ is specific for each setting and mechanism. In order to gain more insight into the incentive properties of PS, Mennle and Seuken (2017*c*) calculated the values of ρ for different settings with three and four objects. Figure 5.1 shows the results of their calculations.

We can see that in addition to the theoretical point of view presented above, these calculations also point to the possibility that the degree of strategyproofness for PS



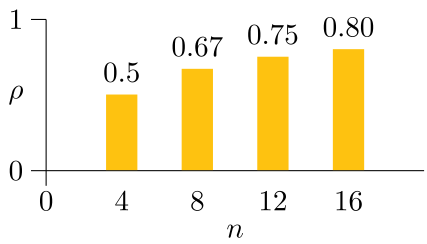


Figure 5.1: Plot of $\rho_{(N,|\hat{O}|,q)}(PS)$ for $|\hat{O}| = 3$ (left) and $m = 4$ (right) objects, for varying numbers of agents n and evenly distributed quotas $q_j = n/|\hat{O}|$, taken from (Mennle and Seuken, 2017*c*).

converges to 1 as the settings get large. This was conjectured by Mennle and Seuken (2017*c*) and the confirmation of this conjecture is the main result of this thesis.

Theorem 1. *Given a fixed set of objects $\hat{O}$, and any $r \in [0,1)$.*

(a) There exists $M \in \mathbb{N}$ such that for all settings $(N, \hat{O}, q)$, with $\min_{j \in M} q_j \geq M$, and

$$\rho_{(N,\hat{O},q)}(PS) \geq r. \tag{5.1}$$

(b) Claim (a) is satisfied for $M \geq x\tilde{D}/\tilde{d}$, where $x \approx 1.76322$ solves $xln(x) = 1$, $\tilde{D} = r^{-\frac{|\hat{O}|}{2}+1}$, $\tilde{d} = \min\{\frac{1}{\sqrt{r}} - 1, 1\}$.

Theorem 1 says that for any setting and any $r \in [0,1)$, the degree of strategyproofness is larger or equal than r if the quotas of all objects are larger than M. The number $M \in \mathbb{N}$ only depends on the chosen r and the number of objects. Note that it is independent of the number of agents in the setting. Nevertheless, the number of agents influences the quotas of the objects, as the demand of the agents must be covered. Therefore, a larger number of agents implies larger quotas of objects, which in turn allows choosing larger bounds r. Hence, as the number of agents gets arbitrarily large, the respective bound r gets arbitrarily close to 1, i.e., converges to 1.

5.2 A Parametric Proof

The major implication of this result is that it leads to a *parametric* proof for the fact that PS is strategyproof in the large.

As mentioned above, Theorem 1 was conjectured by Mennle and Seuken (2017*c*). Based on this conjecture, they showed the following:

Fact 7 (Theorem 3, Statement 3, (Mennle and Seuken, 2017*c*))**.** *Fix a finite set of utility functions* $\{u^1, \ldots, u^K\}$ *and a sequence of settings* $(N^n, \hat{O}^n, q^n)_{n\geq 1}$ *with* $|N^n| = n$, $\hat{O}^n = \hat{O}$, $\sum_{j\in\hat{O}} q_j^n \geq n$ *and* $\min_{j\in\hat{O}} q_j^n \to \infty$ *as* $n \to \infty$*. If the degree of strategyproofness of the mechanism* φ *converges to 1 as* $n \to \infty$*, then* φ *is strategyproof in the large. The converse may not hold.*

In words, this means that for any mechanism φ and any finite set of utility functions, if the settings get large as specified above and the degree of strategyproofness converges to 1 as the number of agents gets arbitrarily large, then the mechanism φ is strategyproof in the large.

Using this fact for PS together with Theorem 1 leads to a short, elegant, parametric proof for the fact that PS is strategyproof in the large.

Fact 8. *The probabilistic serial mechanism is strategyproof in the large.*

Proof: By Theorem 1, we know that the degree of strategyproofness for PS converges to 1 as settings get large. Therefore, by Fact 7, PS is strategyproof in the large.

■

5.3 Other Implications

In addition to the parametric proof presented above, part (b) of Theorem 1 allows us to give bounds for the minimal degree of strategyproofness in a given setting and the maximal necessary quotas of objects for a given setting to ensure r-partial strategyproofness. We look at these two bounds separately.

Firstly, for a fixed number of objects $|\hat{O}|$ and any $r \in [0,1)$, we can give an upper bound M on the minimal quota of objects necessary to guarantee that PS is r-partially strategyproof for any setting with $|\hat{O}|$ objects, independent of the number of agents. For any $r \geq 1/4$ this bound is

$$M(r, |\hat{O}|) = \left\lceil \frac{x}{r^{\frac{|\hat{O}|+1}{2}}(1-\sqrt{r})} \right\rceil, \tag{5.2}$$

where $x \approx 1.76322$ solves $x\ln(x) = 1$. This means, if we would like PS to be r-partially strategyproof for a fixed $r \in [0,1)$ and a fixed setting $(N, \hat{O}, q)$, then we can guarantee that PS is r-partially strategyproof if $\min_{j\in\hat{O}} q_j \geq M(r, |\hat{O}|)$. However, it is possible that PS is r-partially strategyproof for a lower minimal quota of objects.

Secondly, for a fixed number of objects $|\hat{O}|$ and a lower bound of the quotas of objects $M \leq \min_{j\in\hat{O}} q_j$ we get a lower bound for the degree of strategyproofness. This bound can not be expressed analytically in general, as this requires to solve a polynomial of degree

$|\hat{O}| + 1$[2]. However, the bound can be numerically calculated for given settings. Note that the solutions of such a polynomial might be a complex number where the imaginary part is non-zero. In order to get a lower bound for the degree of strategyproofness, we need a real-number solution. While calculating the bounds for different inputs, one could observe that real-number solutions appear for sufficiently large minimal quotas M.

In order to illustrate the quality of these bounds, we calculated them for the settings for which we also know the actual degree of strategyproofness from (Mennle and Seuken, 2017*c*).

The bounds for the settings with three objects and different numbers of agents are shown in figure 5.2.

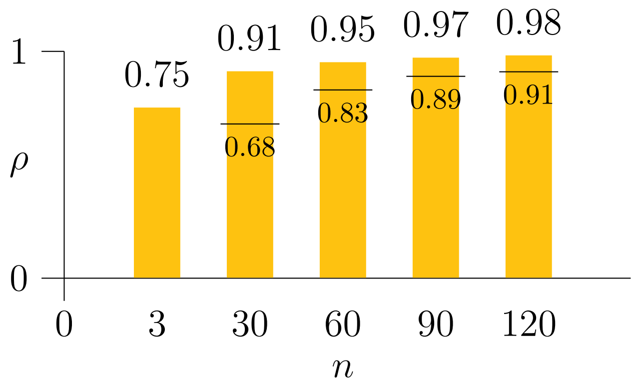


Figure 5.2: Plot of $\rho_{(N,|\hat{O}|,q)}(PS)$ for $|\hat{O}| = 3$ objects, for varying numbers of agents n and evenly distributed quotas $q_j = n/|\hat{O}|$, including the lower bounds for ρ based on part (b) of Theorem 1 (adapted from (Mennle and Seuken, 2017*c*)).

Note that for three agents and a minimal quota of $M = 1$, there is no real-number solution. One can see that the lower bounds of the degree of strategyproofness get closer to the actual value as the number of agents and, therefore, the minimal quota of objects, gets larger.

Conversely, we can calculate the upper bound of the minimal quota of objects to ensure $\rho_{(N,|\hat{O}|,q)}(PS) \geq 0.98$. By evaluating the formula in (5.2) for $r = 0.98$ and $|\hat{O}| = 3$, we get $M = 176$. In the settings from (Mennle and Seuken, 2017*c*) where quotas are uniformly distributed and exactly match the demand of the agents, this would require a maximum of 528 agents to guarantee a degree of strategyproofness of 0.98.

In the case of four objects, the settings considered by (Mennle and Seuken, 2017*c*) did not produce real-number solutions. The first lower bound for the degree of strategyproofness with four agents arises with a minimal quota of 8 per object, which would be the case for 32 agents in the setting of (Mennle and Seuken, 2017*c*). Using $|\hat{O}| = 4$ and $M = 8$ we

[2] The polynomial is $Mr^{(|\hat{O}|+1)/2} - Mr^{|\hat{O}|/2} - x$. A first step to solve it, would be to substitute $r = u^2$, which leads to a polynomial without fractional exponents of degree $|\hat{O}| + 1$.

get a lower bound $0.45 \leq \rho_{(N,|\hat{O}|,q)}(PS)$ for the degree of strategyproofness. In order to ensure a degree of strategyproofness of at least 0.80, the upper bound of the minimal quota of objects is $M = 19$, i.e., 76 agents in terms of the setting from (Mennle and Seuken, 2017*c*).

Finally, these bounds allow more precise strategic advice for agents in a market that uses PS. From Mennle and Seuken (2017*c*) we know that the following advice is applicable for any market that uses a partial strategyproof mechanism:

"They are best off reporting their preferences truthfully as long as their preference intensities for any two objects are sufficiently different; otherwise, if they are close to being indifferent between some objects, then their potential gain from misreporting may be positive but it is limited in the sense of approximate strategyproofness" (Mennle and Seuken, 2017*c*, page 23).

Given that the minimal quota of objects M in the setting is large enough, we can give a lower bound r on the degree of strategyproofness, which allows us to reduce the uncertainty in the term *"sufficiently"*. Therefore, if for any two objects the preference intensities differ by a factor smaller[3] than r, then truthful reporting is definitely a dominant strategy. However, if the intensities differ by a factor larger than r, it may still be a dominant strategy to report truthfully if the factor is sufficiently smaller than 1. This is particularly useful in large markets, as it is infeasible to algorithmically calculate the degree of strategyproofness in large markets.

Note that the potential gain from misreporting is limited since partial strategyproofness implies approximate strategyproofness.

[3]Note that since $r \in [0, 1)$ a smaller factor r implies a larger difference in utility.

6

Tightness of the $URBI(r)$ domain for PS

In this section, we examine if the set of utility functions for which PS is strategyproof is equivalent to $URBI(r)$, in a setting with as many agents as objects and where all objects have unit capacity.

Informally, r-partial strategyproofness says that truthful reporting is a dominant strategy for every agent with a utility function in $URBI(r)$. However, this does not prevent the existence of utility functions outside of $URBI(r)$ which also make truthful reporting a dominant strategy. Nevertheless, it has been shown that $URBI(r)$ is the largest set of utility functions that makes truthful reporting a dominant strategy among all the r-partially strategyproof mechanisms (Mennle and Seuken, 2017*c*). This means that even if there is a utility function outside of $URBI(r)$ that makes truthful reporting a dominant strategy for a given r-partially strategyproof mechanism φ, there exists another r-partially strategyproof mechanism ψ where the same utility function gives rise to a beneficial manipulation.

In the case of PS, which is r-partially strategyproof, Mennle and Seuken (2017*c*) showed that in a setting with three agents and three objects with unit capacity, the set of utility functions that make truthful reporting a dominant strategy is exactly equal to $URBI(r)$. This observation led to the conjecture for any setting where the number of agents is equal to the number of objects and where all objects have unit capacity. It proposes that the set of utility functions for which truthful reporting is a dominant strategy under PS is identical to the set $URBI(r)$, where r is defined by the degree of strategyproofness of PS in the respective setting.

We present the reasoning of Mennle and Seuken (2017*c*) for the case of three agents and three objects with unit capacity. Furthermore, we show that a similar statement holds true for four agents and four objects with unit capacity but does not hold for five agents and five objects with unit capacity.

3-by-3 Settings (adapted from (Mennle and Seuken, 2017*b*): Consider the setting where $N = \{1, 2, 3\}$, $\hat{O} = \{a, b, c\}$ and for all $j \in \hat{O}$, $q_j = 1$. Let the agents have the following preferences:

$$\succ_1 \quad : \quad a \succ b \succ c, \tag{6.1}$$
$$\succ_2 \quad : \quad b \succ a \succ c, \tag{6.2}$$
$$\succ_3 \quad : \quad b \succ c \succ a. \tag{6.3}$$

If all agents report there preferences truthfully under PS we get

$$PS_1 \left(\succ_1, \succ_{2,3}\right) = \left(\frac{3}{4}, 0, \frac{1}{4}\right). \tag{6.4}$$

If agent 1 misreports

$$\succ_1' \quad : \quad b \succ a \succ c, \tag{6.5}$$

we get

$$PS_1 \left(\succ_1', \succ_{2,3}\right) = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right). \tag{6.6}$$

Whether this manipulation is beneficial for agent 1 or not depends on the utility function of agent 1. Let $u_1 \in U_{\succ_1}$ be the utility function of agent 1. Then the expected gain in utility from this misreport is

$$u_1 \left(PS_1 \left(\succ_1', \succ_{2,3}\right)\right) - u_1 \left(PS_1 \left(\succ_1, \succ_{2,3}\right)\right) = -\frac{1}{4}u_1(a) + \frac{1}{3}u_1(b) - \frac{1}{12}u_1(c). \tag{6.7}$$

This is weakly negative if and only if

$$\frac{u_1(a) - u_1(c)}{u_1(b) - u_1(c)} \leq \frac{3}{4}. \tag{6.8}$$

Note that this is exactly the condition that has to hold such that $u_1 \in UBRI(3/4)$. Moreover, we know from the calculations of Mennle and Seuken (2017*c*) that for this setting, PS has a degree of strategyproofness of $\rho_{(N,\hat{O},q)}(PS) = 3/4$ (See figure 5.1 in section 5).

If there exists a utility function $u_1' \notin UBRI(3/4)$ such that truthful reporting is a dominant strategy, it must also give no rise for misreporting in the situation presented above. Therefore, it has to satisfy the condition derived in (6.8) to not give rise for any misreport, but violate the condition that defines $UBRI(3/4)$. However, as argued above, these two conditions are identical to each other, therefore, there does not exist such a utility function u_1'. Hence, in the setting with three agents and three objects with unit capacity, the set of utility functions that make truthful reporting a dominant strategy is exactly $URBI(3/4)$.

4-by-4 Settings: Next, we show that in the setting where $N = \{1,2,3,4\}$, $\hat{O} = \{a,b,c,d\}$ and for all $j \in \hat{O}$, $q_j = 1$, the set of utility functions that make truthful reporting a dominant strategy is exactly $URBI(1/2)$. We already know that PS has a degree of strategyproofness of $\rho_{(N,\hat{O},q)}(PS) = 1/2$ (See figure 5.1 in section 5).

The approach will be similar to the one in the 3-by-3 setting. We will present preferences of agents and derive the conditions for a weakly negative expected utility gain from certain manipulations. These conditions together will coincide with the definition of $URBI(1/2)$ in the 4-by-4 setting.

First, consider the following preferences:

$$\succ_1 \quad : \quad a \succ b \succ c \succ d, \tag{6.9}$$
$$\succ_2 \quad : \quad b \succ c \succ d \succ a, \tag{6.10}$$
$$\succ_3 \quad : \quad c \succ a \succ b \succ d, \tag{6.11}$$
$$\succ_4 \quad : \quad c \succ b \succ d \succ a, \tag{6.12}$$

and the following misreport of agent 1:

$$\succ_1' \quad : \quad b \succ a \succ c, \succ d. \tag{6.13}$$

Then we get the following assignments under PS:

$$PS_1\left(\succ_1, \succ_{2,3,4}\right) = \left(\frac{3}{4}, 0, 0, \frac{1}{4}\right), \tag{6.14}$$
$$PS_1\left(\succ_1', \succ_{2,3,4}\right) = \left(\frac{1}{2}, \frac{1}{2}, 0, 0\right). \tag{6.15}$$

Let $u_1 \in U_{\succ_1}$ again be the utility function of agent 1. Then the expected gain in utility from this misreport is

$$u_1\left(PS_1\left(\succ_1', \succ_{2,3,4}\right)\right) - u_1\left(PS_1\left(\succ_1, \succ_{2,3,4}\right)\right) = -\frac{1}{4}u_1(a) + \frac{1}{2}u_1(b) - \frac{1}{4}u_1(d). \tag{6.16}$$

This is weakly negative if and only if

$$\frac{u_1(a) - u_1(d)}{u_1(b) - u_1(d)} \leq \frac{1}{2}. \tag{6.17}$$

Next, consider the preferences

$$\succ_1 = \succ_2 \quad : \quad a \succ b \succ c \succ d, \tag{6.18}$$
$$\succ_3 = \succ_4 \quad : \quad a \succ c \succ d \succ b, \tag{6.19}$$
$$\tag{6.20}$$

and the following misreport of agent 1:

$$\succ_1' \quad : \quad a \succ c \succ b, \succ d, \tag{6.21}$$

Then we get the following assignments under PS:

$$PS_1\left(\succ_1, \succ_{2,3,4}\right) = \left(\frac{1}{4}, \frac{1}{2}, 0, \frac{1}{4}\right), \tag{6.22}$$

$$PS_1\left(\succ_1', \succ_{2,3,4}\right) = \left(\frac{1}{4}, \frac{1}{3}, \frac{1}{3}, \frac{1}{12}\right). \tag{6.23}$$

The expected gain in utility from this misreport is

$$u_1\left(PS_1\left(\succ_1', \succ_{2,3,4}\right)\right) - u_1\left(PS_1\left(\succ_1, \succ_{2,3,4}\right)\right) = -\frac{1}{6}u_1(b) + \frac{1}{3}u_1(c) - \frac{1}{6}u_1(d). \tag{6.24}$$

Therefore, the expected gain in utility is weakly negative if and only if

$$\frac{u_1(c) - u_1(d)}{u_1(b) - u_1(d)} \leq \frac{1}{2}. \tag{6.25}$$

Observe that the conditions in (6.17) and (6.25) are equivalent to the conditions that $u_i \in URBI(1/2)$. Therefore, for any utility function that is not in $URBI(1/2)$, there exists a situation where misreporting is beneficial. Hence, in the setting of four agents and four objects with unit capacity, the set of utility functions for which truthful reporting is a dominant strategy is exactly $URBI(1/2)$.

5-by-5 Settings: In the setting of five agents and five objects with unit capacity, i.e., $N = \{1, 2, 3, 4, 5\}$, $\hat{O} = \{a, b, c, d, e\}$ and $\forall j \in \hat{O}$, $q_j = 1$, PS has a degree of strategyproofness of $\rho_{(N,\hat{O},q)}(PS) = 1/2$, which can be determined via the algorithm used by Mennle and Seuken (2017*b*). In contrast to the previous cases, the set of utility functions for which truthful reporting is a dominant strategy in this setting is strictly larger than $URBI(1/2)$.

Consider the following utility function:

$$u_1 = (7.99, 4, 2, 1, 0). \tag{6.26}$$

Note that this utility functions violates $URBI(1/2)$, since

$$\frac{u_1(a) - u_1(e)}{u_1(b) - u_1(e)} = \frac{4}{7.99} > \frac{1}{2}. \tag{6.27}$$

Using an implementation of PS, we could evaluate the potential utility gain for every possible preference profile and every possible manipulation for the given setting by iterating through all of them.[4] This computation showed that for the utility function u_1 in (6.26) there is no beneficial misreport, despite the fact that it violates $URBI(1/2)$.

Thereby, we disprove the conjecture that the set of utility functions for which PS is strategyproof coincides with $URBI(r)$ in settings with equal numbers of agents and objects and where the objects have unit capacity.

[4]The implementation makes use of the fact that PS is neutral and anonymous, i.e., the mechanism is independent of the names of objects and the names of agents. This allows to consider a subset of all possible preference profiles, as the allocations for the remaining profiles can be inferred by renaming the agents and objects of considered profiles.

7

Proof of the Theorem

We will prove Theorem 1 in three steps: Firstly, we will prove Lemma 2 which states that there is a basis for the set of utility functions that satisfy uniformly relatively bounded indifference and are consistent with a given preference order. Moreover, we will adjust Fact 1 form (Kojima and Manea, 2010) to work with this basis. Fact 1 states that the probabilistic serial mechanism (PS) is strategyproof in large settings for finitely many strict utility functions. Note that the base utility functions from Lemma 2 are not all strict. Together, these statements will allow us to prove Lemma 3, which established that for any $r \in [0, 1)$ there is a lower bound on the quotas of objects such that, if the bound is satisfied, PS is r-locally partial strategyproof.

Secondly, by Fact 9, r-local partial strategyproofness implies r^2-partial strategyproofness. With this, we can extend the claim of Lemma 3 to partial strategyproofness in Lemma 5.

Finally, Lemma 5 directly implies Theorem 1 as it holds for all $r \in [0, 1)$. Figure 7.1 illustrates these steps.

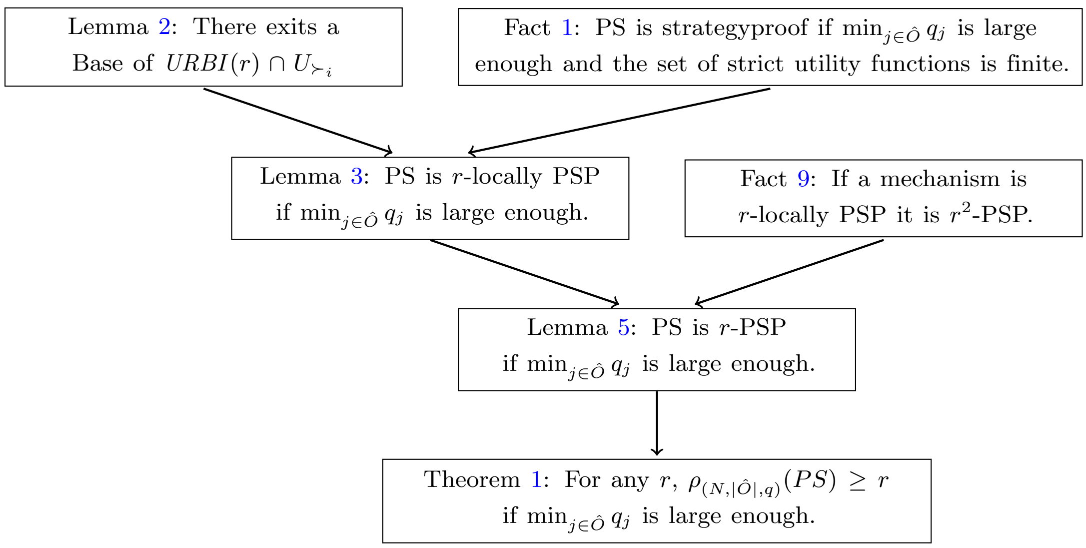


Figure 7.1: Visual representation of the proof idea, PSP = partial strategyproofness.

7.1 Basis of $URBI(r) \cap U_{\succ_i}$

In order to find a basis for a given set, it is necessary that this set is convex. Therefore, we first establish that the set $URBI(r) \cap U_{\succ_i}$ is convex by proving the following lemma:

Lemma 1. *For any $r \in [0,1]$ and any given preference order $\succ_i \in \mathcal{P}$ the set $URBI(r) \cap U_{\succ_i}$ is convex.*

Proof: Fix a preference order $\succ_i \in \mathcal{P}$ and $r \in [0,1]$. For any utility functions $u \in URBI(r) \cap U_{\succ_i}$ and any two objects $a, b \in \hat{O}$ we have by definition

$$r \cdot \left(u_i(a) - \min_{j \in \hat{O}} u_i(j) \right) \geq u_i(b) - \min_{j \in \hat{O}} u_i(j). \tag{7.1}$$

In order to show that $URBI(r) \cap U_{\succ_i}$ is convex, we need to show that any convex combination u_i^* of utility functions in $URBI(r) \cap U_{\succ_i}$ is again in $URBI(r) \cap U_{\succ_i}$. Formally,

$$\forall u_i', u_i'' \in URBI(r) \cap U_{\succ_i} \text{ and } \forall \mu \in [0,1], \quad u_i^* = \mu u_i' + (1-\mu) u_i'' \in URBI(r) \cap U_{\succ_i}. \tag{7.2}$$

Since u_i' and u_i'' are in $U_{\succ_i}$ their convex combination u_i^* is naturally again in $U_{\succ_i}$. Moreover, $\min u_i^* = \mu \min u_i' + (1-\mu) \min u_i''$ since u_i' and u_i'' are minimal for the same object. Therefore, we only need to show that the conditions for $URBI(r)$ are still satisfied.

For any two objects $a, b \in \hat{O}$ where $u_i^*(a) > u_i^*(b)$ we get:

$$r \left(u_i^*(a) - \min_{j \in \hat{O}} u_i^*(j) \right) = r \left(\left(\mu u_i'(a) + (1-\mu) u_i''(a) \right) - \min_{j \in \hat{O}} \left(\mu u_i'(j) + (1-\mu) u_i''(j) \right) \right) \tag{7.3}$$

$$= r\mu \left(\mu u_i'(a) - \min_{j \in \hat{O}} u_i'(j) \right) + r(1-\mu) \left(\mu u_i''(a) - \min_{j \in \hat{O}} u_i''(j) \right) \tag{7.4}$$

$$\geq \mu \left(\mu u_i'(b) - \min_{j \in \hat{O}} u_i'(j) \right) + (1-\mu) \left(\mu u_i''(b) - \min_{j \in \hat{O}} u_i''(j) \right) \tag{7.5}$$

$$= u_i^*(b) - \min_{j \in \hat{O}} u_i^*(j). \tag{7.6}$$

■

Since we now know that $URBI(r) \cap U_{\succ_i}$ is convex, it is possible to find a basis such that the convex set $URBI(r) \cap U_{\succ_i}$ is contained in the positive linear hull of the basis. This means that for any $u_i \in URBI(r) \cap U_{\succ_i}$ there exists a set of utility functions $\mathcal{B} \ni b_i^s$ such that u_i can be expressed as a linear combination $u_i(j) = \sum_{s=1}^{|\hat{O}|-1} \mu_s b_i^s(j)$, where for all s, $\mu_s \geq 0$.

In order to formally present this basis we need to introduce the *rank* of an object for a given agent. Let $rank_i(j)$ be the position of object j in the preference order $\succ_i$, e.g., if $\succ_i$: $a \succ b \succ c$, then $rank_i(a) = 1$, $rank_i(b) = 2$. Now we can state the following lemma:

Lemma 2. *For any given set of objects $\hat{O}$, any preference order $\succ_i \in \mathcal{P}$ and any $r \in [0,1]$, we have that any utility function $u_i \in URBI(r) \cap U_{\succ_i}$ can be expressed as linear combination $u_i(j) = \sum_{s=1}^{|\hat{O}|-1} \mu_s b_i^s(j)$, where for all $s \in \{1, \ldots |\hat{O}| - 1\}$, $\mu_j \geq 0$, and for all $b_i^s \in \mathcal{B}(r, |\hat{O}|, \succ_i)$,*

$$b_i^s(j) = \begin{cases} \frac{1}{r^{|\hat{O}|-rank_i(j)-s}} & \text{if } rank_i(j) \leq |\hat{O}| - 1 \text{ and } |\hat{O}| - rank_i(j) - s \geq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{7.7}$$

Note that not every linear combination of basis elements $b_i^s(j)$ will produce a valid utility functions that is consistent with $\succ_i$. For a better understanding of how this basis looks like, consider the following example.

Example 3. *For a fixed a set of objects $\hat{O} = \{a, b, c, d\}$, a preference order $\succ_i$: $a \succ b \succ c$ and $r \in [0,1]$, we would have:*

$$\mathcal{B}(r, 4, \succ_i) = \{b_i^1, b_i^2, b_i^3\} = \left\{ \left(\frac{1}{r^2}, \frac{1}{r}, 1, 0\right), \left(\frac{1}{r}, 1, 0, 0\right), (1, 0, 0, 0) \right\} \tag{7.8}$$

where $b_i^1 = (b_i^1(a), b_i^1(b), b_i^1(c), b_i^1(d)) = \left(\frac{1}{r^2}, \frac{1}{r}, 1, 0\right)$.

Proof: Firstly, we show that any linear combination of base utility functions that is consistent with $\succ_i$. Since $\succ_i$ is a strict preference order, the only base utility function that is consistent with $\succ_i$ is $b^1(j)$. Note that every base utility function $b_i^s(j)$ is strictly decreasing for the first few objects with respect to $\succ_i$ and is constant afterwards. Therefore, every linear combination $u_i(j) = \sum_{s=1}^{|\hat{O}|-1} \mu_s b_i^s(j)$, where for all $s \in \{1, \ldots |\hat{O}| - 1\}$, $\mu_s \geq 0$ and $\mu_1 > 0$, is consistent with $\succ_i$.

Secondly, we show that every base utility functions satisfies $URBI(r)$. Note that we can simplify the definition of $URBI(r)$, since $\min_{j \in \hat{O}} b_i^s(j) = 0$. Therefore, we get

$$\forall a, b \in \hat{O}, \text{ where } b_i^s(a) > b_i^s(b) \Rightarrow r b_i^s(a) \geq b_i^s(b). \tag{7.9}$$

For any consecutive pair of objects $a, b \in \hat{O}$, where $b_i^s(a) \geq b_i^s(b)$ we have one of three cases:

- **Case $\boldsymbol{b_i^s(a) = 0}$:** Then $b_i^l sb) = 0$ as $b_i^s(a) \geq b_i^s(b)$ and $r \cdot 0 \geq 0$ trivially holds for any r.
- **Case $\boldsymbol{b_i^s(a) > 0}$ and $\boldsymbol{b_i^s(b) = 0}$:** Then the condition $r \cdot b_i^s(a) \geq 0$ trivially holds for any $r \in [0,1]$.

- **Case $b_i^s(a) > 0$ and $b_i^s(b) > 0$:** Then we have that $rank_i(a) + 1 = rank_i(b)$. Therefore, we have that

$$rb_i^s(a) = r\frac{1}{r^{|\hat{O}|-rank_i(a)-s}} = \frac{1}{r^{|\hat{O}|-(rank_i(a)+1)-s}} = \frac{1}{r^{|\hat{O}|-rank_i(b)-s}} = b_i^s(b). \quad (7.10)$$

  Thus $rb_i^s(a) \geq b_i^s(b)$ holds for any $r \in [0, 1]$.

Therefore, every base utility functions satisfies $URBI(r)$. Moreover, this establishes that any linear combination $u_i(j) = \sum_{s=1}^{|\hat{O}|-1} \mu_s b_i^s(j)$ satisfies $URBI(r)$.

Finally we need to prove that the positive linear hull of $\mathcal{B}(r, |\hat{O}|, \succ_i)$ contains all of $URBI(r) \cap U_{\succ_i}$. In order to show that $URBI(r)$ is contained in the linear hull of the base utility functions, we show that we have as many linear independent base elements as there are degrees of freedom in $URBI(r)$.

The inequalities that define $URBI(r)$ need to hold for any two objects $a, b \in \hat{O}$, where $a \succ_i b$. However, due to the transitive nature of weak inequalities "$\geq$" and the monotone decrease of the utility function with respect to the preference order $\succ_i$, it is sufficient to ensure the inequality on successive neighbours with respect to the preference order $\succ_i$. Therefore, if there are $|\hat{O}|$ objects, then there are $|\hat{O}| - 1$ inequalities that need to hold for a strict preference order $\succ_i \in \mathcal{P}$. This is exactly the number of base elements defined in the Lemma, as l runs from 1 to $|\hat{O}| - 1$.

It remains to be shown that the utility functions $b_i^s \in \mathcal{B}(r, |\hat{O}|, \succ_i)$ are linearly independent, i.e., no b_i^s can be created as a linear combination of other elements of $\mathcal{B}(r, |\hat{O}|, \succ_i)$. Observe that the element b_i^s has s trailing zeros. This is due to the condition $|\hat{O}| - rank_i(j) - s \geq 0$ in the definition of b_i^s. With every increase of the counting variable s the condition breaks one rank earlier and thus leads to an additional zero. Hence, if we remove the k^{th} base utility function from the set, there is no linear combination of the remaining base utility functions to create a utility function with a strictly positive utility for the object at rank k and a utility of zero for all objects with rank larger than k. Thereby, the set of utility functions $\mathcal{B}(r, |\hat{O}|, \succ_i)$ is linearly independent.

With the condition that $\mu_1 > 0$, we have that the linear hull of $\mathcal{B}(r, |\hat{O}|, \succ_i)$ contains all of $URBI(r) \cap U_{\succ_i}$. This completes the proof of this Lemma.

■

7.2 Local Partial Strategyproofness

The definition of local partial strategyproofness is similar to the definition of partial strategyproofness, however, insead of all possible misreports only the misreports in the neighbourhood $\mathcal{N}_{\succ_i}$ are considered. Recall that neighbourhood $\mathcal{N}_{\succ_i}$ is the set of strict preference orders that differ from $\succ_i$ by only one swap of consecutively ranked objects with respect to $\succ_i$.

Definition 11 (Local Partial Strategyproofness). *Given a setting* $(N, \hat{O}, q)$ *and a bound* $r \in [0,1]$, *a mechanism* φ *is* r-local partial strategyproof (in the setting $(N, \hat{O}, q)$) *if, for all agents* $i \in N$, *all preference profiles* $(\succ_i, \succ_{N\setminus\{i\}}) \in \mathcal{P}^N$, *all misreports in the neighbourhood* $\succ_i' \in \mathcal{N}_{\succ_i}$ *and all utility functions* $u_i \in U_{\succ_i} \cap URBI(r)$, *we have*

$$u_i\left(\varphi\left(\succ_i, \succ_{N\setminus\{i\}}\right)\right) \geq u_i\left(\varphi\left(\succ_i', \succ_{N\setminus\{i\}}\right)\right). \tag{7.11}$$

In order to prove Lemma 5 in section 7.4, we will use the following fact from Mennle and Seuken (2017*a*):

Fact 9 (Theorem 1,(Mennle and Seuken, 2017*a*)). *Given a setting* $(N, \hat{O}, q)$, *if a mechanism* φ *is* r*-locally partially strategyproof, then it is* r^2*-partially strategyproof.*

This fact means, that in order to show that a mechanism φ is r-partial strategyproof it suffices to show that the mechanism is $\sqrt{r}$-locally partially strategyproof for the more demanding bound $\sqrt{r} > r$.

7.3 Lemma for Local Partial Strategyproofness

The following lemma adapts the result from (Kojima and Manea, 2010) for local partial strategyproofness. Informally, for a given setting and a given bound r, PS is r-locally partially strategyproof for sufficiently large quotas of objects. Moreover, the lemma provides a lower bound on the quotas of objects such that the claim is guaranteed to hold.

Lemma 3 (adapted form (Kojima and Manea, 2010)). *For any setting* $(N, \hat{O}, q)$ *and for any* $r \in [0, 1)$

(a) there exists $M \in \mathbb{N}$ *such that if* $\forall j \in \hat{O} : q_j \geq M$, *then PS is* r*-local partial strategyproof. Formally, for all utility functions* $b_i^s \in \mathcal{B}(r, |\hat{O}|, \succ_i)$, *we have*

$$b_i^s\left(PS(\succ_i, \succ_{N\setminus\{i\}})\right) \geq b_i^s\left(PS(\succ_i', \succ_{N\setminus\{i\}})\right) \tag{7.12}$$

$\forall i \in N, \succ_i \in \mathcal{P}, \succ_i' \in \mathcal{N}_{\succ_i}, \forall \succ_{N\setminus\{i\}} \in \mathcal{P}^{N\setminus\{i\}}$.

(b) Claim (a) is satisfied for $M \geq x\tilde{D}/\tilde{d}$, *where* $x \approx 1.76322$ *solves* $x ln(x) = 1$, $\tilde{D} = r^{-|\hat{O}|+2}$, $\tilde{d} = \min\{\frac{1}{r} - 1, 1\}$.

In order to prove Lemma 3, we rely on facts and notations, which are directly adopted from (Kojima and Manea, 2010). We now introduce the facts and notations necessary to prove part (a) of Lemma 3.

In analogy to the example 1 in the introduction, Kojima and Manea (2010) use the concept of *eating functions*. An *eating function* is a right-continuous function

$e_i : [0,1] \to O$, where $e_i(t) \in O$ is the object that agent i is eating at time t. Right-continuous means that $\forall t \in [0,1)$, $\exists \varepsilon > 0$, such that $e_i(t) = e_i(t')$, $\forall t' \in [t, t+\varepsilon)$.

For an eating function e The *number of agents that eat object a at a given time t* is denoted by $n_a(t,e) = |\{i \in N \mid e_t) = a\}|$.

The *share* of an object a that is eaten away by time t is denoted by $v_a(t,e) = \int_0^t n_a(s,e)ds$. Note that $v_a(\cdot, e)$ is right-continuous.

For a preference profile $\succ \in \mathcal{P}^N$, we denote by $e^{\succ}$ the eating function generated by the symmetric simultaneous eating algorithm if all agents report truthfully. If agent i misreports $\succ_i'$ instead of $\succ_i$, then we denote the resulting preference profile by $\succ' = (\succ_i', \succ_{N\setminus\{i\}})$.

In order to compare the length of the time intervals in which agent i consumes a different object when reporting $\succ_i$ than when reporting $\succ_i'$, the following functions are used:

$$\beta(t) = \int_0^t \mathbf{1}_{e_i^{\succ'}(s) \succ_i e_i^{\succ}(s)} ds, \quad \gamma(t) = \int_0^t \mathbf{1}_{e_i^{\succ}(s) \succ_i e_i^{\succ'}(s)} ds, \quad \delta(t) = \beta(t) + \gamma(t), \tag{7.13}$$

where for any logical expression p, $\mathbf{1}_p = 1$ if p is true and $\mathbf{1}_p = 0$ otherwise. The function $\beta(t)$ returns the summed up length of all time intervals up to t, where agent i strictly prefers the object it eats under $\succ$ (truthful report) to the object it would eat under $\succ'$ (misreport). Analogously, $\gamma(t)$ returns the summed up length of all time intervals up to t, where agent i strictly prefers the object it eats under $\succ'$ (misreport) to the object it would eat under $\succ$ (truthful report). The summed up length of all time intervals up to t, where the object eaten by agent i differs from $\succ$ to $\succ'$ is measured in $\delta(t)$. Consider the following example:

Example 4. *Fix $N = \{1,2,3\}$, $\hat{O} = \{a,b,c\}$ and $q_j = 1$ for all $j \in \hat{O}$. The agents have the following preferences:*

$$\succ = ((a \succ_1 b \succ_1 c), (b \succ_3 c \succ_2 a), (b \succ_3 a \succ_3 c)). \tag{7.14}$$

Then we get the assignment probabilities for agent 1: $PS_1(\succ) = (3/4, 0, 1/4)$. If agent 1 misreports $\succ_1' = b \succ_1' c \succ_1' a$, then we get $PS_1(\succ') = (1/6, 1/3, 1/2)$. Figure 7.2 illustrates the corresponding eating functions. Moreover, the figure shows the thick time intervals where the eating functions do agree with each other. The sum of the length of these intervals is equal to $\delta(1) = \beta(1) + \gamma(1)$. Furthermore, the length of the longer, thick, left interval is equal to $\gamma(1)$ and the length of the shorter, thick interval is equal to $\beta(1)$.[5]

Furthermore, Kojima and Manea (2010) define the set of objects $\{a_1, a_2, \ldots, a_{\bar{l}}\}$, which

[5] Note that the time interval where the misreport is beneficial for agent 1 is significantly shorter than the time interval where the misreport is detrimental for agent 1 ($\beta(1) < \gamma(1)$). The difference in the length of these intervals is central to the proof of Fact 1 from (Kojima and Manea, 2010).

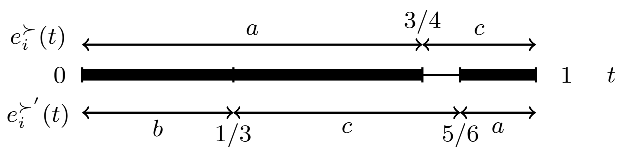


Figure 7.2: Illustration of Example 4, inspired from (Kojima and Manea, 2010).

are $\succ_i$-prefered to $e_i^{\succ}(t)$ at some time t, under $\succ'$. Formally,

$$\{a_1, a_2, \ldots, a_{\bar{l}}\} = \{a \in \hat{O} \mid \exists t \in [0, 1), a = e_i^{\succ'}(t) \succ_i e_i^{\succ}(t)\}. \tag{7.15}$$

The set is labeled such that $a_1 \succ_i' a_2 \succ_i' \cdots \succ_j' a_{\bar{l}}$. For every $l = 1, 2, \ldots, \bar{l}$, define T_l as the first time when a_l is consumed under $e_i^{\succ'}$ and is $\succ_i$-prefered to the consumption under $e_i^{\succ}$, i.e.,

$$T_l = \inf_{t \in [0,1)} \{a_l = e_i^{\succ'}(t) \succ_i e_i^{\succ}(t)\}. \tag{7.16}$$

Note that $0 < T_1 < T_2 < \cdots < T_{\bar{l}} < 1$. As a technical notation convention, set $T_0 = 0$ and $T_{\bar{l}+1} = 1$.

Let k denote the number of proper objects $a \in \hat{O}$ that are $\succ_i$-preferred to the null object, $k = |\{a \in \hat{O} \mid a \succ_i \phi\}|$. Note that $\bar{l} \leq k$, since for all l, $a_l = e_i^{\succ'}(t) \succ_i e_i^{\succ}(t) \succeq_i \phi$.

Finally, we define the following auxiliary eating function: Fix a preference profile $\succ$ and an agent i. Let $\succ'$ be the preference profile which is composed of a misreport of agent i and the original preference profile for all other agents. Then, define

$$\bar{e}_i(t) = \begin{cases} e_i^{\succ}(t) & \text{if } \succ_i(t) = \succ_i'(t) \\ \phi & \text{otherwise.} \end{cases} \tag{7.17}$$

Under $\bar{e}_j$, every agent $j \neq i$ eats its most preferred object among the ones still available, considering the consumption of agent i according to $\bar{e}_i$. Note that through the possible changes of i's eating behaviour, the eating functions $\bar{e}, e_i^{\succ}$ and $e_i^{\succ}$ may differ from each other.

After the introduction of these notations, we present the lemmas from (Kojima and Manea, 2010), which are the crucial building blocks to prove their main result. These lemmas are denoted by "KM-Lemma". The lemmas build up on each other and some will be used to show our adjusted version of the result of Kojima and Manea (2010).

KM-Lemma 1 (Lemma 1, (Kojima and Manea, 2010)). *For all $t \in [0, 1]$ and $a \in \hat{O}$,*

$$v_a(t, e^{\succ}) \geq v_a(t, \bar{e}) \text{ and } v_a(t, e^{\succ'}) \geq v_a(t, \bar{e}). \tag{7.18}$$

KM-Lemma 2 (Lemma 2, (Kojima and Manea, 2010))**.** *For all* $t \in [0,1]$*,*

$$v_\phi(t, e^{\succ}) - v_\phi(t, \bar{e}) \geq -\delta(t). \tag{7.19}$$

KM-Lemma 3 (Lemma 3, (Kojima and Manea, 2010))**.** *For all* $t \in [0,1]$ *and* $a \in \hat{O}$*,*

$$v_a(t, e^{\succ}) - v_a(t, \bar{e}) \leq \delta(t). \tag{7.20}$$

KM-Lemma 4 (Lemma 4, (Kojima and Manea, 2010))**.** *For all* $t \in [0,1]$ *and* $a \in \hat{O}$*,*

$$v_a(t, e^{\succ}) - v_a(t, e^{\succ'}) \leq \delta(t). \tag{7.21}$$

KM-Lemma 5 (Lemma 5, (Kojima and Manea, 2010))**.** *For all* $l = 1, \ldots, \bar{l}$*,*

$$\beta(T_{l+1}) - \beta(T_l) \leq \frac{\delta(T_l)}{q_{a_l}}. \tag{7.22}$$

KM-Lemma 6 (Lemma 6, (Kojima and Manea, 2010))**.** *If* $q_a \geq M$ *for all* $a \in \hat{O}$*, then*

$$\beta(T_{l+1}) - \beta(T_l) \leq \frac{\gamma(1)}{M}\left(1 + \frac{1}{M}\right)^{l-1}, \quad \forall l = 0, 1, \ldots \bar{l}. \tag{7.23}$$

As elaborated in the proof of part (a) of Lemma 3 below, we want to apply these KM-Lemmas for local manipulations that involve objects with positive utility. However, since the base utility functions of $URBI(r) \cap U_{\succ_i}$ can also assign a utility of zero to multiple objects, we need a slightly adjusted version of KM-Lemma 6 in order for the proof to work.

Let $\tilde{t}$ be the time when $b_i^s(e_i^{\succ}(\tilde{t})) = b_i^s(e_i^{\succ'}(\tilde{t})) = 0$ holds for the first time, i.e.,

$$\tilde{t} = \inf\{t' \in [0,1] \mid b_i^s(e_i^{\succ}(t')) = b_i^s(e_i^{\succ'}(t')) = 0\}. \tag{7.24}$$

Note that by definition of $\tilde{t}$ we have that $\forall t > \tilde{t}, b_i^s(e_i^{P'}(t)) = b_i^s(e_i^{P'}(t')) = 0$.

We define $\tilde{l} \leq \bar{l}$ such that

$$T_{\tilde{l}-1} < \tilde{t} \leq T_{\tilde{l}}. \tag{7.25}$$

Lemma 4 (adapted from Lemma 6 (Kojima and Manea, 2010))**.** *If* $q_a \geq M$ *for all* $a \in \hat{O}$*, then*

$$\beta(T_{l+1}) - \beta(T_l) \leq \frac{\gamma(\tilde{t})}{M}\left(1 + \frac{1}{M}\right)^{l-1}, \quad \forall l = 0, 1, \ldots \tilde{l} - 1. \tag{7.26}$$

Note that the difference to KM-Lemma 6 is the time at which the function γ is evaluated and the maximal index of l for which the lemma holds.

Proof: [adapted from Kojima and Manea (2010)] The proof of this lemma will be done by induction on l.

For $l = 0$ the induction hypothesis holds trivially as $\beta(T_0) = \beta(T_1) = 0$.
For $l = 1$, we use KM-Lemma 5 to get that $\beta(T_2) - \beta(T_1) \leq \frac{\delta(T_1)}{q_{a_1}}$, where $\delta(T_1) = \gamma(T_1)$ as $\beta(T_1) = 0$. Furthermore, since γ is monotonically increasing, $T_1 \leq \tilde{t}$ and $q_{a_l} \geq M$ by assumption, we get

$$\beta(T_2) - \beta(T_1) \leq \frac{\delta(T_1)}{q_{a_1}} = \frac{\gamma(T_1)}{q_{a_1}} \leq \frac{\gamma(\tilde{t})}{q_{a_1}} \leq \frac{\gamma(\tilde{t})}{M} = \frac{\gamma(\tilde{t})}{M} \cdot \left(1 + \frac{1}{M}\right)^0. \tag{7.27}$$

Thereby, the induction hypothesis holds for $l = 1$.

Now, let $l \geq 2$. Suppose that the induction hypothesis holds for $l = 0, 1, \ldots, l-1$. We prove that it also holds for l. We want to apply KM-Lemma 5, thus, we first consider the term $\delta(T_l)$:

$$\delta(T_l) = \gamma(T_l) + \beta(T_l) \leq \gamma(\tilde{t}) + \beta(T_l). \tag{7.28}$$

Note, that even if $l = \tilde{l} - 1$, $T_l = T_{\tilde{l}-1} \leq \tilde{t}$ by the definition of $\tilde{l}$, therefore, $\gamma(T_l) \leq \gamma(\tilde{t})$.

We can write $\beta(T_l)$ using a telescope sum and then apply the induction hypothesis:

$$\beta(T_l) = \sum_{g=1}^{l-1} \beta(T_{g+1}) - \beta(T_g) \leq \frac{\gamma(\tilde{t})}{M} \sum_{g=1}^{l-1} \left(1 + \frac{1}{M}\right)^{g-1}. \tag{7.29}$$

Note that this last sum is a partial sum of a geometric series. Therefore, we can use the formula $\sum_{k=0}^{n} q^k = \frac{q^{n+1}-1}{q-1}$ and get

$$\frac{\gamma(\tilde{t})}{M} \sum_{g=1}^{l-1} \left(1 + \frac{1}{M}\right)^{g-1} = \frac{\gamma(\tilde{t})}{M} \cdot \frac{(1 + \frac{1}{M})^{l-1} - 1}{(1 + \frac{1}{M}) - 1} = \gamma(\tilde{t}) \left(\left(1 + \frac{1}{M}\right)^{l-1} - 1\right). \tag{7.30}$$

If we combine (7.28) with (7.30), then we get

$$\delta(T_l) \leq \gamma(\tilde{t}) + \gamma(\tilde{t}) \left(\left(1 + \frac{1}{M}\right)^{l-1} - 1\right) = \gamma(\tilde{t}) \left(1 + \frac{1}{M}\right)^{l-1}. \tag{7.31}$$

Since $q_{a_l} \geq M$ by assumption, KM-Lemma 5 and (7.31) imply that

$$\beta(T_{l+1}) - \beta(T_l) \leq \frac{\gamma(\tilde{t})}{M} \left(1 + \frac{1}{M}\right)^{l-1}, \tag{7.32}$$

which completes the proof of the induction step.

■

Now that we have proven the adjusted version of KM-Lemma 6, we can prove part (a) of Lemma 3.

Proof of Part (a):
From Lemma 1 we know that the set of utility functions $URBI(r) \cap U_{\succ_i}$ is convex and by Lemma 2, every utility function $u_i \in URBI(r) \cap U_{\succ_i}$, can be written as linear combination of $b_i^s \in \mathcal{B}(r, |\hat{O}|, \succ_i)$. Therefore, in order to show r-local partial strategyproofness of PS it suffices to show that PS is strategyproof for the utility functions $b_i^s \in \mathcal{B}(r, |\hat{O}|, \succ_i)$.

Fix a setting $(N, \hat{O}, q)$ and $r \in [0, 1)$. Note that each utility functions $b_i^s \in \mathcal{B}(r, |\hat{O}|, \succ_i)$ is decreasing with respect to a fixed preference order $\succ_i \in \mathcal{P}$. Furthermore, the set of objects $\hat{O}$ can be split into a part of objects with strictly decreasing utilities with respect to $\succ_i$ and a part of objects with utility zero. Consider the following example as an illustration only:

$$b_i^3 = \left(b_i^3(a), b_i^3(b), b_i^3(c), b_i^3(d), b_i^3(e), b_i^3(f)\right) = \left(\frac{1}{r^2}, \frac{1}{r}, 1, 0, 0, 0\right) \in \mathcal{B}(r, 6, \succ_i), \tag{7.33}$$

the set of objects $\hat{O}$ can be split into

$$\hat{O}_>(b_i^3) = \{a, b, c\} \text{ and } \hat{O}_0(b_i^3) = \{d, e, f\}. \tag{7.34}$$

Note that this split can be done for any $b_i^s \in \mathcal{B}(r, |\hat{O}|, \succ_i)$.

Fix any agent $i \in N$, a preference order $\succ_i$ and any utility function $b_i^s \in \mathcal{B}(r, |\hat{O}|, \succ_i)$. We simplify the notation of the splits of the set of objects as follows: $\hat{O}_> = \hat{O}_>(b_i^s)$ and $\hat{O}_0 = \hat{O}_0(b_i^s)$.

Since $\hat{O} = \hat{O}_> \cup \hat{O}_0$, proving Lemma 3 is equivalent to showing local partially strategyproofness for all $b_i^s \in \mathcal{B}(r, |\hat{O}|, \succ_i)$ on $\hat{O}_>$, $\hat{O}_0$ and across these sets. The remainder of this proof is, therefore, split into three parts: (i) Considering local manipulations on $\hat{O}_>$, (ii) considering the only possible local manipulation across $\hat{O}_>$ and $\hat{O}_0$ and (iii) considering local manipulations on $\hat{O}_0$.

Part (i): Local manipulations on $\hat{O}_0$:
According to Fact 5 the probabilistic serial mechanism is r-partially strategyproof for some $r > 0$. Furthermore, by Fact 4 a mechanism is partially strategyproof if and only if it is swap monotonic and upper invariant. Therefore, PS is swap monotonic and upper invariant.

If there is only one object in $\hat{O}_0$, then there are no local manipulations that have to be checked within this set. Therefore, assume that $|\hat{O}_0| \geq 2$. PS is locally partially strategyproof on $\hat{O}_0$ if the following holds:

$$b_i^s\left(PS(\succ_i, \succ_{N\setminus\{i\}})\right) \geq b_i^s\left(PS(\succ_i', \succ_{N\setminus\{i\}})\right) \tag{7.35}$$

for all $\succ_i \in \mathcal{P}$, for all misreports $\succ_i' \in \mathcal{N}_{\succ_i}$ where only swaps of objects in $\hat{O}_0$ are considered and for all possible reports of other agents $\succ_{N\setminus\{i\}} \in \mathcal{P}^{N\setminus\{i\}}$. Pick two consecutive objects

$d, e \in \hat{O}_0$ and swap there order in $\succ_i$ to create the misreport $\succ_i'$. By the upper invariance of PS we know that the assignment probabilities for objects in the upper contour set $U(d, \succ_i)$ will not change. Therefore, no utility gain or loss can originate from objects that agent i prefers over d, especially from no object in $\hat{O}_>$. Moreover, possible changes in the assignment probability of objects d, e can not lead to utility gain or loss, as they have a utility of zero. Similarly, any changes for objects less preferred than e by i can also not lead to utility gain or loss. Thereby, swapping consecutive objects of $\hat{O}_0$ is never beneficial.

Part (ii): Local Manipulation across $\hat{O}_>$ and $\hat{O}_0$:
The set $\hat{O}_>$ is defined such that it contains all objects with strictly positive utility. Therefore, the swap of consecutive objects c, d across $\hat{O}_>$ and $\hat{O}_0$ is a swap of objects where $b_i^s(c) = 1$ and $b_i^s(d) = 0$. According to the upper invariance of PS, the assignment probabilities for objects in the upper contour set $U(c, \succ_i)$ will not change. The misreport $\succ_i'$ is created by swapping the order of c and d in $\succ_i$. Based on Fact 4, PS is swap monotonic. Therefore, the assignment probabilities of c and d can either stay as they are or the assignment probability of c decreases and the assignment probability of d increases. However, since $b_i^s(d) = 0$, such a manipulation can not increase, the expected utility of the assignment.

Part (iii): Local manipulations on $\hat{O}_>$.
In order to show local-partial strategyproofness of PS on $\hat{O}_>$, we use an adaptation of the result of (Kojima and Manea, 2010).

As mentioned above, the eating function $e_i^{\succ}(t)$ returns which object is consumed by agent i for any time $t \in [0, 1]$. Furthermore, as the time t runs from 0 to 1, the time an agent is eating a certain object a is equivalent to the assignment probability of this object to agent i under PS. Therefore, we can calculate the expected utility of PS by integrating the eating function over time:

$$b_i^s\left(PS(\succ_i)\right) = \int_0^1 b_i^s\left(e_i^{\succ}(t)\right) dt. \tag{7.36}$$

This allows us to write the expected utility *loss* from misreporting $\succ_i'$ instead of $\succ_i$ as

$$b_i^s\left(PS(\succ_i, \succ_{N\setminus\{i\}})\right) - b_i^s\left(PS(\succ_i', \succ_{N\setminus\{i\}})\right) = \int_0^1 b_i^s\left(e_i^{\succ}(t)\right) - b_i^s\left(e_i^{\succ'}(t)\right) dt. \tag{7.37}$$

Let $\tilde{t}$ be the time when $b_i^s(e_i^P(\tilde{t})) = b_i^s(e_i^{P'}(\tilde{t})) = 0$ holds for the first time, i.e.,

$$\tilde{t} = \inf\{t' \in [0,1] \mid b_i^s(e_i^P(t')) = b_i^s(e_i^{P'}(t')) = 0\}. \tag{7.38}$$

The fact that $\forall t > \tilde{t}$, we have $b_i^s(e_i^{P'}(t)) = b_i^s(e_i^{P'}(t')) = 0$, implies that $\int_{\tilde{t}}^1 b_i^s\left(e_i^{\succ}(t)\right) - b_i^s\left(e_i^{\succ'}(t)\right) dt = 0$. Hence,

$$\int_0^1 b_i^s\left(e_i^{\succ}(t)\right) - b_i^s\left(e_i^{\succ'}(t)\right) dt = \int_0^{\tilde{t}} b_i^s\left(e_i^{\succ}(t)\right) - b_i^s\left(e_i^{\succ'}(t)\right) dt. \tag{7.39}$$

Next, we want to show that this expected utility loss from misreporting is strictly larger than zero, which is identical to showing that the expected utility gain is weakly negative.

In their proof, Kojima and Manea (2010) used the following inequality to bound the utility loss from below:

$$\int_0^1 u_i\left(e_i^{\succ}(t)\right) - u_i\left(e_i^{\succ'}(t)\right) dt \geq d\gamma(1) - D\beta(1), \tag{7.40}$$

where

$$D = \max_{a \succeq_i b \succeq_i \phi} u_i(a) - u_i(b) \quad d = \min_{a \succ_i b, a \succeq_i \phi} u_i(a) - u_i(b). \tag{7.41}$$

This means that D is the maximal utility difference and d the minimal utility difference of u_i. As the authors only allow utility functions consistent with strict preference profiles, $d > 0$ will always be true. However, in our case, where we use the base utility functions $b_i^s \in \mathcal{B}(r, |\hat{O}|, \succ_i)$, the smallest utility difference is zero. This means that this bound could not be used to show that the utility difference is bounded from below by zero. The only thing left on the right side of the inequality in (7.40) is $-D\gamma(1)$, which is clearly smaller than zero. Nevertheless, with the fact that the utility does not change after $\tilde{t}$ and by the definition of $\tilde{t}$, it follows that $d > 0$ for all $t < \tilde{t}$.

Therefore, we re-define d and D as $\tilde{d}$ and $\tilde{D}$ as follwos: Let $e, f \in \hat{O}$, $|\hat{O}| \geq 2$ and $b_i^s \in \mathcal{B}(r, |\hat{O}|, \succ_i)$, then

$$\tilde{D} = \max_{e \succeq_i f \succeq_i \phi} b_i^s(e) - b_i^s(f) = r^{-|\hat{O}|+2}, \tag{7.42}$$

$$\tilde{d} = \min_{e \succ_i f, e \succeq_i \phi, b_i^s(e) > 0} b_i^s(e) - b_i^s(f) = \min\left\{\frac{1}{r} - 1, 1\right\}. \tag{7.43}$$

By the definition of $\tilde{t}$ there does not exist a $t < \tilde{t}$ such that $b_i^s\left(e_i^{\succ}(t)\right) = b_i^s\left(e_i^{\succ'}(t)\right) = 0$. Therefore, the following inequality is valid as well:

$$\int_0^{\tilde{t}} b\left(e_i^{\succ}(t)\right) - b\left(e_i^{\succ'}(t)\right) dt \geq \tilde{d}\gamma(\tilde{t}) - \tilde{D}\beta(\tilde{t}). \tag{7.44}$$

Now, we use Lemma 4 to get an estimate for $\beta(\tilde{t})$. By summing up the segments $\beta(T_{g+1}) - \beta(T_g)$ until $T_g \geq \tilde{t}$, we can give an upper bound for $\beta(\tilde{t})$. Then, we apply Lemma 4 to estimate these segments and sum them up using the formula for geometric series, while considering that $\beta(T_1) = 0$.

$$\beta(\tilde{t}) \leq \sum_{g=0}^{\tilde{l}-1} \beta(T_{g+1}) - \beta(T_g) \leq \sum_{g=0}^{\tilde{l}-1} \frac{\gamma(\tilde{t})}{M} \cdot \left(1 + \frac{1}{M}\right)^{\tilde{l}} = \gamma(\tilde{t}) \left(\left(1 + \frac{1}{M}\right)^{\tilde{l}} - 1\right). \tag{7.45}$$

Since $\tilde{l} \leq k \leq |\hat{O}|$, where k is the number of proper objects in $\hat{O}$, we also get

$$\gamma(\tilde{t}) \left(\left(1 + \frac{1}{M}\right)^{\tilde{l}} - 1\right) \leq \gamma(\tilde{t}) \left(\left(1 + \frac{1}{M}\right)^{k} - 1\right) \leq \gamma(\tilde{t}) \left(\left(1 + \frac{1}{M}\right)^{|\hat{O}|} - 1\right). \tag{7.46}$$

Therefore, we get

$$\int_0^{\tilde{t}} b_i^s \left(e_i^{\succ}(t)\right) - b_i^s \left(e_i^{\succ'}(t)\right) dt \geq \gamma(\tilde{t}) \left(\tilde{d} - \tilde{D}\left(\left(1 + \frac{1}{M}\right)^{|\hat{O}|} - 1\right)\right), \tag{7.47}$$

which is non-negative if

$$M \geq \frac{1}{\left(\frac{\tilde{d}}{\tilde{D}} + 1\right)^{1/|\hat{O}|} - 1}. \tag{7.48}$$

This concludes part (iii) and part (a) of Lemma 3, as we found a M such that PS is partially strategyproof under the given conditions.

■

Finally, for part (b) of Lemma 3 we need one more lemma from (Kojima and Manea, 2010). Define

$$\tilde{\Lambda} = \frac{\gamma(\tilde{t})}{M} \left(1 + \frac{1}{M}\right)^{k-1}. \tag{7.49}$$

KM-Lemma 7 (adapted from Lemma 7, (Kojima and Manea, 2010))**.** *Suppose $q_a \geq M$ for all $a \in \hat{O}$. Then for all $a \in \hat{O}$ and $t \leq T_{\tilde{l}}$ with $t + \tilde{\Lambda} \leq \tilde{t}$,*

$$v_a\left(t, e^{\succ}\right) = q_a \Rightarrow v_a\left(t + \tilde{\Lambda}, e^{\succ'}\right) = q_a. \tag{7.50}$$

This KM-Lemma says that if an object runs out at time t under truthful reporting, then the same object will have run out by $t + \tilde{\Lambda}$ for the preference profile where one agent misreported.

Note that the KM-Lemma 7 is almost identical to the original, however, we redefined Λ in (7.49) to depend on $\gamma(\tilde{t})$ instead of $\gamma(1)$. We denote this redefinition by $\tilde{\Lambda}$[6].

[6]The adjusted $\tilde{\Lambda}$, does not affect the proof presented by Kojima and Manea (2010).

Proof of Part (b):
As in part (a) it suffices to consider the utility functions $b_i^s \in \mathcal{B}(r, |\hat{O}|, \succ_i)$, since every utility function $u_i \in URBI(r) \cap U_{\succ_i}$, can be written as linear combination of $b_i^s \in \mathcal{B}(r, |\hat{O}|, \succ_i)$.

In order to prove part (b), we use KM-Lemma 7 to obtain a tighter estimate of the possible utility loss, which can then be transformed into the claimed estimate for M.

Assume that $q_a \geq M$ for all $a \in \hat{O}$. The consequence of KM-Lemma 7 that $v_{a_{\tilde{l}-1}}\left(T_{\tilde{l}-1} + \tilde{\Lambda}, e^{\succ'}\right) = q_{a_{\tilde{l}-1}}$ if $T_{\tilde{l}-1} + \tilde{\Lambda} \leq \tilde{t}$ leads to

$$\forall t > \min\{T_{\tilde{l}-1} + \tilde{\Lambda}, \tilde{t}\}, \quad b_i^s\left(e_i^{\succ'}(t)\right) \leq b_i^s\left(e_i^{\succ}(t)\right). \tag{7.51}$$

For technical purposes, we extend $e_i^{\succ}$ such that for all $t \in [-\tilde{\Lambda}, 0)$, $e_i^{\succ}(t) = e_i^{\succ}(0)$. Therefore,

$$\forall t \in [0, \tilde{t}], \quad b_i^s\left(e_i^{\succ'}(t)\right) \leq b_i^s\left(e_i^{\succ}(t - \tilde{\Lambda})\right). \tag{7.52}$$

If we apply this to the expected utility difference, we obtain

$$\begin{aligned}
& b_i^s(PS(\succ)) - b_i^s(PS(\succ')) && (7.53)\\
&= \int_0^{\tilde{t}} b_i^s\left(e_i^{\succ}(t)\right) - b_i^s\left(e_i^{\succ'}(t)\right) dt && (7.54)\\
&= \int_0^{\tilde{t}} \max\left\{0, b_i^s\left(e_i^{\succ}(t)\right) - b_i^s\left(e_i^{\succ'}(t)\right)\right\} dt + \int_0^{\tilde{t}} \min\left\{0, b_i^s\left(e_i^{\succ}(t)\right) - b_i^s\left(e_i^{\succ'}(t)\right)\right\} dt && (7.55)\\
&\geq \tilde{d}\gamma(\tilde{t}) + \int_0^{\tilde{t}} \min\left\{0, b_i^s\left(e_i^{\succ}(t)\right) - b_i^s\left(e_i^{\succ}(t - \tilde{\Lambda})\right)\right\} dt, && (7.56)
\end{aligned}$$

where $\tilde{d} = \min\{\frac{1}{r} - 1, 1\}$. In equation (7.56), we estimated the positive parts of the utility difference, which sum up to $\gamma(\tilde{t})$ using the minimal positive utility difference $\tilde{d}$. Moreover, we made use of (7.52) to substitute $b\left(e_i^{\succ'}(s)\right)$.

Next, since $b_i^s\left(e_i^{\succ}(t)\right) - b_i^s\left(e_i^{\succ}(t - \tilde{\Lambda})\right) \leq 0$ for all $t \in [0, \tilde{t}]$, we can drop the minimum. Furthermore, we use basic arithmetic operations for integrals to transform the equation.

$$\begin{aligned}
(7.56) &= \tilde{d}\gamma(\tilde{t}) + \int_0^{\tilde{t}} b_i^s\left(e_i^{\succ}(t)\right) - b_i^s\left(e_i^{\succ}(t - \tilde{\Lambda})\right) dt && (7.57)\\
&= \tilde{d}\gamma(\tilde{t}) + \int_0^{\tilde{t}} b_i^s\left(e_i^{\succ}(t)\right) ds - \int_{-\tilde{\Lambda}}^{\tilde{t}-\tilde{\Lambda}} b_i^s\left(e_i^{\succ}(t)\right) dt && (7.58)\\
&= \tilde{d}\gamma(\tilde{t}) + \int_{\tilde{t}-\tilde{\Lambda}}^{\tilde{t}} b_i^s\left(e_i^{\succ}(t)\right) dt - \int_{-\tilde{\Lambda}}^{0} b_i^s\left(e_i^{\succ}(t)\right) dt && (7.59)
\end{aligned}$$

$$= \tilde{d}\gamma(\tilde{t}) + \int_{-\tilde{\Lambda}}^{0} b_i^s\left(e_i^{\succ}(t)\right) - b_i^s\left(e_i^{\succ}(t+\tilde{t})\right) dt \tag{7.60}$$

$$\geq \tilde{d}\gamma(\tilde{t}) - \tilde{D}\tilde{\Lambda}, \tag{7.61}$$

where $\tilde{D} = r^{-|\hat{O}|+2}$, given $|\hat{O}| \geq 2$. In the last integral, we know that the utility differences of the objects consumed in the interval $[-\tilde{\Lambda}, 0)$ compared to the objects consumed in $[\tilde{t}-\tilde{\Lambda}, \tilde{t})$ are surely less than the maximal utility difference $\tilde{D}$ and use this as an estimate.

Substituting the definition of $\tilde{\Lambda}$ we get

$$b_i^s(PS(\succ)) - b_i^s(PS(\succ')) \geq \tilde{d}\gamma(\tilde{t}) - \tilde{D}\tilde{\Lambda} = \tilde{d}\gamma(\tilde{t}) - \tilde{D}\frac{\gamma(\tilde{t})}{M}\left(1 + \frac{1}{M}\right)^{k-1}, \tag{7.62}$$

which simplifies to

$$b_i^s(PS(\succ)) - b_i^s(PS(\succ')) \geq \frac{\tilde{d}\gamma(\tilde{t})}{M}\left(M - \frac{\tilde{D}}{\tilde{d}}\left(1 + \frac{1}{M}\right)^{k-1}\right). \tag{7.63}$$

Supposed that $M \geq x\tilde{D}/\tilde{d}$, where $x \approx 1.76322$ solves $x\ln(x) = 1$. Let $\boldsymbol{e} \approx 2.71828$ denote Euler's number. Note that $\tilde{D}/\tilde{d} \geq k$, where k is the number of proper objects in $\hat{O}$. With $M \geq x\tilde{D}/\tilde{d} \geq xk$ we get

$$\left(1 + \frac{1}{M}\right)^{k-1} < \left(1 + \frac{1}{xk}\right)^{k} = \left(\left(1 + \frac{1}{xk}\right)^{xk}\right)^{1/x} < \boldsymbol{e}^{1/x}. \tag{7.64}$$

As $x = \boldsymbol{e}^{1/x}$ and by using $M = x\tilde{D}/\tilde{d}$ and (7.64), it follows that

$$b_i^s(PS(\succ)) - b_i^s(PS(\succ')) \geq \frac{\tilde{d}\gamma(\tilde{t})}{M}\left(x\frac{\tilde{D}}{\tilde{d}} - \frac{\tilde{D}}{\tilde{d}}\boldsymbol{e}^{1/x}\right) = 0. \tag{7.65}$$

Hence the claim of part (a) holds if $M \geq x\tilde{D}/\tilde{d}$.

This concludes the proof of part (b) and, therefore, the proof of Lemma 3.

■

7.4 Lemma for Partial Strategyproofness

As a next step, we extend the claim of Lemma 3 for local partial strategyproofness to partial strategyproofness, using Fact 9. We get the following lemma:

Lemma 5. *For any setting* $(N, \hat{O}, q)$ *and for any* $r \in [0, 1)$

(a) there exists $M \in \mathbb{N}$ *such that if* $\forall j \in \hat{O} : q_j \geq M$*, then PS is* r*-partially strategyproof. Formally,* $\forall u_i \in URBI(r) \cap U_{P_i}$*,* $\forall i \in N, \succ_i' \in \mathcal{P}, \forall \succ_{N\backslash\{i\}} \in \mathcal{P}^{N\backslash\{i\}}$*, we have*

$$u_i \left(PS \left(\succ_i, \succ_{N\backslash\{i\}} \right) \right) \geq u_i \left(PS \left(\succ_i', \succ_{N\backslash\{i\}} \right) \right). \tag{7.66}$$

(b) Claim (a) is satisfied for $M \geq x\tilde{D}/\tilde{d}$*, where* $x \approx 1.76322$ *solves* $x ln(x) = 1$*,* $\tilde{D} = r^{-\frac{|\hat{O}|}{2}+1}$*,* $\tilde{d} = \min\{\frac{1}{\sqrt{r}} - 1, 1\}$*.*

Proof: Fix a setting $(N, \hat{O}, q)$ and a $r \in [0, 1)$. Define $r_{loc} = \sqrt{r}$, note that for all $r \in [0, 1)$, $\sqrt{r} > r$ and $\sqrt{r} \in [0, 1)$. Therefore, by Lemma 3 we know that in the given setting and with $r_{loc} \in [0, 1)$ there exists a $M \in \mathbb{N}$ such that if $\forall j \in \hat{O} : q_j \geq M$, PS is r_{loc}-locally partial strategyproof. Futhermor, we know that this holds for $M = x\tilde{D}/\tilde{d}$, where $x \approx 1.76322$ solves $x\ln(x) = 1$, $\tilde{D} = r_{loc}^{-|\hat{O}|+2}$, $\tilde{d} = \min\{\frac{1}{r_{loc}} - 1, 1\}$.

Part (a): From Fact 9 we know that if a mechanism is r_{loc}-locally partially strategyproof, it is $(r_{loc})^2$-partially strategyproof. By definition of r_{loc} we have $(r_{loc})^2 = \sqrt{r}^2 = r$. Thereby, in the given setting and with $r \in [0, 1)$ there exists a $M \in \mathbb{N}$ such that if $\forall j \in \hat{O} : q_j \geq M$, PS is r-partially strategyproof.

Part (b): If we substitute r_{loc} for its definition $\sqrt{r}$ we get the according formulas for $\tilde{D}$ and $\tilde{d}$, i.e.,

$$\tilde{D} = r_{loc}^{-|\hat{O}|+2} = r^{-\frac{|\hat{O}|}{2}+1}, \tag{7.67}$$

$$\tilde{d} = \min\{\frac{1}{r_{loc}} - 1, 1\} = \min\{\frac{1}{\sqrt{r}} - 1, 1\}. \tag{7.68}$$

■

7.5 Proof of Theorem 1

Finally, Theorem 1 is directly implied from Lemma 5.

Theorem 1. *Given a fixed set of objects $\hat{O}$, and any $r \in [0, 1)$.*

(a) There exists $M \in \mathbb{N}$ such that for all settings $(N, \hat{O}, q)$, with $\min_{j \in M} q_j \geq M$, and

$$\rho_{(N,\hat{O},q)}(PS) \geq r. \tag{7.69}$$

(b) Claim (a) is satisfied for $M \geq x\tilde{D}/\tilde{d}$, where $x \approx 1.76322$ solves $xln(x) = 1$, $\tilde{D} = r^{-\frac{|\hat{O}|}{2}+1}$, $\tilde{d} = \min\{\frac{1}{\sqrt{r}} - 1, 1\}$.

Proof:

Part (a): The degree of strategyproofness for PS is defined as

$$\rho_{(N,\hat{O},q)}(PS) = \max\{r \in [0,1] \mid PS \text{ is } r\text{-partially strategyproof in } (N, \hat{O}, q)\}. \tag{7.70}$$

From Lemma 5 we know that for any given $r \in [0, 1)$ we can find a $M \in \mathbb{N}$ such that if $\forall j \in \hat{O} : q_j \geq M$, PS is r-partially strategyproof for all agents. Hence, $\rho_{(N,\hat{O},q)}(PS) \geq r$.

Part (b): The estimate for M follows directly from Lemma 5.

■

8

Discussion

In this section we will discuss two things concerning the proof of Theorem 1: Firstly, we discuss why Lemma 5 in the proof of the theorem is not directly implied by the result from Kojima and Manea (2010). Secondly, we discuss an alternative approach to prove the theorem, which turned out to not be feasible due to a false intuition about misreports under PS.

8.1 Lemma 5 and Kojima and Manea (2010)

Recall Fact 2, which is a corollary in (Kojima and Manea, 2010). It states that for any fixed setting with a finite set of proper objects $\hat{O}$ and a fix finite set of utility functions $\mathcal{U}$, there exists a lower bound M on the minimal quota of objects such that PS is strategyproof in the given setting.

In Lemma 5, we show that for any finite setting $(N, \hat{O}, q)$ and any $r \in [0, 1)$ there exists a lower bound M on the minimal quotas of objects such that PS is r-partially strategyproof in the given setting.

The key difference in the conditions of these two claims is that in Lemma 5, we do not assume a finite set of utility functions. Instead, the lemma ensures that truth-telling is a dominant strategy for any of the infinitely many utility functions that satisfies $URBI(r)$. Therefore, no matter how big the finite set of utility functions for the corollary in Kojima and Manea (2010) is, it will never cover all possible utility functions.

Furthermore, applying the result of Kojima and Manea (2010) (Fact 2) on the finite set of utility functions that form a basis of $URBI(r) \cap U_{\succ_i}$ for every agent i and every possible preference order $\succ_i$, does not work either. The statement from Kojima and Manea (2010) assumes that the utility functions are consistent with strict preference orders, however, as we see in Lemma 2, not all base utility functions are strictly decreasing. This condition can not simply be dropped in Fact 2, since this will break the formula for the lower bound on the object quotas M. Recall that this formula is $M = xD/d$, where $x \approx 1.76322$ solves $x\ln(x) = 1$, $D = \max_{a \succeq_i b \succeq_i \phi} u_i(a) - u_i(b)$, and $d = \min_{a \succ_i b, a \succeq_i \phi} u_i(a) - u_i(b)$. If we calculate d for a not strictly decreasing base utility function we get zero, which prevents the existence of M.

Hence, the special structure of the base of $URBI(r) \cap U_{\succ_i}$ has to be taken into account. In order to proof Lemma 5, we considered local partial strategyproofness in Lemma 3, which allowed us to resolve the issue where multiple objects have utility zero. Furthermore, we adjusted the result of Kojima and Manea (2010) such that it can applied for misreports that involve objects from the strictly decreasing part of the base utility functions.

8.2 A False Intuition

Under PS, one might have the intuition that whatever the truthful ranking is, it is never beneficial to report an object with utility zero at any other rank but last. This means if there is an object with utility zero (a zero-object), you will always want to report it last. The idea behind this intuition is, that the time you need to eat this zero-object could be used to eat an object with positive utility, which would still increase the over-all utility, even if the gained utility is small.

If this intuition is true, then we could directly proof Lemma 5 about partial strategyproofness in large markets as follows. For any given r, we would need to show that for any utility function in $URBI(r)$ PS is strategyproof for all preference profiles and all possible misreports. We would do this by showing the claim for the base utility functions $b_i^s \in \mathcal{B}(r, |\hat{O}|, \succ_i)$. To cover all possible misreports, note that any misreport can be created in two steps: first, misreport the orders of the objects with positive utility with respect to b_i^s, second, create the desired misreport by swapping the zero-objects accordingly. For the first step, misreporting any object with positive utility is not beneficial by the adjusted result from (Kojima and Manea, 2010) if the minimal quota of objects is large enough. The second step can then be covered by the introduced intuition if it is true.

However, it turns out that this intuition is wrong. There are settings where reporting a zero-object above another object with positive utility can be beneficial under PS. Consider the following counterexample for this intuition.

Let $N = \{1, 2, 3, 4\}$, $\hat{O} = \{a, b, c, d\}$ and $q_j = 1$ for all objects $j \in \hat{O}$. The true preference order of agent 1 is $\succ_1$: $a \succ b \succ c \succ d$ with a utility function $u_1 = (4, 2, 1, 0)$, therefore, object d is a zero-object. Note that $u_1 \in URBI(1/2)$, moreover, it is a base utility function for $URBI(1/2) \cap U_{\succ_1}$. Consider the situation where agents report the following preferences:

$$\succ_1' \quad : \quad b \succ c \succ a \succ d \tag{8.1}$$

$$\succ_2 \quad : \quad b \succ d \succ c \succ a \tag{8.2}$$

$$\succ_3 \quad : \quad c \succ b \succ d \succ a \tag{8.3}$$

$$\succ_4 \quad : \quad d \succ c \succ b \succ a. \tag{8.4}$$

Here, agent 1 misreported $\succ_1'$: $b \succ c \succ a \succ d$. We get

$$PS_1\left(\succ_1', \succ_{2,3,4}\right) = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}, 0\right), \tag{8.5}$$

which yields an expected utility of 2.25 for agent 1.

According to our intuition, any misreport where the zero-object d is not reported last should result in a lower expected utility than the misreport $\succ_i'$. Now consider that agent 1 misreports $\succ_1''$: $b \succ d \succ a \succ c$, i.e., he swaps the zero-object d with object c. We get

$$PS_1\left(\succ_1'', \succ_{2,3,4}\right) = \left(\frac{1}{3}, \frac{1}{2}, 0, \frac{1}{6}\right), \tag{8.6}$$

which yields an expected utility of 2.33 for agent 1, which is larger than 2.25.

Moreover, if we consider the base utility function $u_1' = (1, 0, 0, 0)$, we see that even swapping the two zero-objects c and d can yield a gain in utility. This happens because the sequence in which agent 1 eats the probability shares of the objects influences the eating schedule of the other agents. Note that the intuition holds for three agents and three objects, where each object has unit capacity.

9

Conclusion

In this thesis, we considered the probabilistic serial mechanism (PS) for assignment problems. We saw that while it is ordinal efficient, it is not strategyproof. Nevertheless, PS satisfies r-partial strategyproofness. Partial strategyproofness is a parametric incentive concept, where 1-partial strategyproofness is equal to normal strategyproofness (Mennle and Seuken, 2017*c*).

As our main result, we have shown that the degree of strategyproofness for PS converges to 1 in large markets. We achieved this by showing that for any finite setting and any $r \in [0, 1)$ there exists a minimal quota of objects such that the degree of strategyproofness for PS in the given setting is at least as large as r. This implies convergence because as the markets get large, the quotas of objects also have to get large in order to meet the demand of the agents. However, the result does not depend on the number of agents participating in the mechanism but only on the minimal quota of objects.

This result allowed for an elegant, parametric proof of the fact that PS is strategyproof in the large. Moreover, we have seen that the condition on the minimal quota of objects can be used to give upper bounds on the necessary quota of each object to guarantee a certain degree of strategyproofness for a given setting. Furthermore, given the quotas of objects is large enough, the condition can also be used to give a lower bound on the degree of strategyproofness in a given setting. Also, these bounds allow us to give better advice for agents participating in PS, based on the number of objects, their quotas and the utility function of the agent.

In addition to this, we considered settings where the number of agents equals the number of objects and every object has unit capacity. In these settings, we showed that the subset of utility functions for which the probabilistic serial mechanism is strategyproof is not identical to the subset of utility functions that satisfy uniformly relatively bounded indifference. However, this is true for three agents and three objects with unit capacity, as well as for four agents and objects. Whether this condition is satisfied for larger settings is a matter of future research.

These results deepen our understanding of the incentives of PS in large markets beyond the results for finitely many utility functions from (Kojima and Manea, 2010). Our results are valid for all possible utility functions. Future research may make use of the proof idea to show similar convergence results for different mechanisms, where a result for large markets exists.

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A

Appendix

A.1 Content of the Enclosed CD

- Abstract.txt *(Abstract in English)*
- Zusfsg.txt *(Abstract in German)*
- Masterarbeit.pdf *(Thesis)*

List of Figures