Parametric Optimization and Variational Problems Involving Polyhedral Multifunctions

Diethard Klatte

University of Zurich

diethard.klatte/at/uzh.ch

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In honor of Jiři Outrata's 70th birthday

Thanks for cooperation to Bernd Kummer, Humboldt University Berlin

Tribute to a founder of PARAOPT

František Nožička (1918-2004)

His centenary will be in 2018

Introduction

Given a (m, n)-matrix A, consider the linear program

$$\min_x \ a^\top x \quad \text{s.t.} \quad Ax \le b \tag{1}$$

with parameters $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. Define

$$\begin{aligned}
\Psi(a,b) &= \operatorname{argmin}_x\{a^\top x \mid Ax \le b\} && \text{(optimal solution set)}\\
\varphi(a,b) &= \min_x\{a^\top x \mid Ax \le b\} && \text{(optimal value)}\\
S(a,b) &= \left\{(x,\lambda) \mid \begin{array}{l} A^\top \lambda = -a,\ \lambda \ge 0, \\ Ax \le b,\ \lambda^\top (Ax - b) = 0 \end{array}\right\} && \text{(KKT solution set)}
\end{aligned}$$

Obviously,

- $\operatorname{dom} \Psi = \operatorname{dom} S$ is a polyhedral convex set, while
- $\operatorname{gph} S = \{(a,b,x,\lambda) \mid (x,\lambda) \in S(a,b)\}$ and $\operatorname{gph} \Psi$ (projection) are finite unions of polyhedral convex sets.

Definition. (Robinson '76, '79) A multifunction $\Gamma : \mathbb{R}^s \to \mathbb{R}^r$ is called **polyhedral** if gph Γ is a union of finitely many polyhedral convex sets.

Theorem 1. (Robinson '76, '81) If $\Gamma : \mathbb{R}^s \to \mathbb{R}^r$ is polyhedral, then there is a constant $\varrho > 0$ such that for each $\bar{p}$ and some $\varepsilon = \varepsilon(\bar{p}) > 0$,

$$\Gamma(p) \subset \Gamma(\bar{p}) + \varrho \|p - \bar{p}\| B \quad \forall p \in B(\bar{p}, \varepsilon), \tag{2}$$

i.e., some upper Lipschitz property holds with uniform constant.
Notes: $\Gamma(p)$ may be empty. (2) $\Rightarrow$ either $\bar{p} \in \operatorname{dom} \Gamma$ or $\bar{p} \notin \operatorname{cl} \operatorname{dom} \Gamma$.
(2) implies calmness of Γ.

The optimal set map of (1) is in general not continuous, cf. e.g.,

$$\min \varepsilon x \text{ s.t. } 0 \leq x \leq 1, \qquad \text{i.e., } \Psi(\varepsilon) = \begin{cases} \{1\} & \text{if } \varepsilon < 0 \\ [0,1] & \text{if } \varepsilon = 0 \\ \{0\} & \text{if } \varepsilon > 0. \end{cases}$$

The proof of Theorem 1 makes use of a result by Walkup, Wets '69:

If $\Gamma : \mathbb{R}^s \to \mathbb{R}^r$ is graph-convex and polyhedral, then Γ is Lipschitzian on $\mathrm{dom}\,\Gamma$ w.r. to the Pompeiu-Hausdorff metric, i.e.,

$$\exists \varrho > 0 : \quad d_H(\Gamma(p), \Gamma(p')) \le \varrho \|p - p'\| \quad \forall p, p' \in \mathrm{dom}\,\Gamma. \tag{3}$$

which essentially relies on **Hoffman's Lemma** (Hoffman '52):

Given a (m, n)-matrix A and norms $\|\cdot\|_\alpha$, $\|\cdot\|_\beta$ in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. Consider $b \mapsto M(b) = \{x \mid Ax \le b\}$. Then

$$\exists \lambda_{\alpha\beta} > 0 : \ \mathrm{dist}_\beta(x, M(b)) \le \lambda_{\alpha\beta} \|(Ax - b)_+\|_\alpha \ \ \forall x \in \mathbb{R}^n \ \forall b \in \mathrm{dom}\, M.$$

For explicit Hoffman constants $\lambda_{\alpha\beta}$, see e.g. Robinson '73, Mangasarian '81, Mangasarian, Shiau '87, Li '93 (sharp bound), Kl, Thiere '96.

Consequences for the parametric LP (1) (Robinson '81)

(i) Both the KKT map S and the optimal set map Ψ have the upper Lipschitz property (2).

(ii) The value function φ is Lipschitzian on bounded subsets of $\operatorname{dom}\Psi$.

(ii) also follows from Nožička et al. '74 who show (via partition into local stability sets): φ is continuous and piecewise quadratic.

Remark 1. Let $\Gamma : \mathbb{R}^s \to \mathbb{R}^r$, and let $D \subset \operatorname{dom}\Gamma$ be convex. If Γ is "pointwise Lipschitz" on D w.r. to d_H and with global constant ϱ, i.e.,

$$\boxed{\exists \varrho > 0} \quad \forall \bar{p} \in D\, \exists \varepsilon > 0 : \; d_H(\Gamma(p), \Gamma(\bar{p})) \le \varrho \, \|p - \bar{p}\| \;\; \forall p \in D \cap B(\bar{p}, \varepsilon),$$

then Γ is Lipschitzian on D w.r. to d_H with constant ϱ.

Proof via a "standard trick": (finite) open covering of any segment $[b^1, b^2] \subset D$. (Robinson '81, Kl '84, cf. also Outrata et al. '98)

Content of this talk

1. Upper Lipschitz stability in quadratic optimization

2. Further Lipschitz stability concepts and basic models

3. Lipschitz stability under constraint non-degeneracy

4. Lipschitz stability for degenerate constraints

5. Concluding remarks

1. Upper Lipschitz stability in quadratic optimization

The above Lipschitz properties for the parametric LP (1) carry over to parametric QP

$$\min \ \tfrac{1}{2}x^\top Qx + a^\top x \ \text{ s.t. } \ Ax \le b, \qquad (a,b) \text{ varies}, \tag{4}$$

with given (m,n)-matrix A and symmetric (n,n)-matrix Q. Let

- $\Psi(a,b)$ ($\varphi(a,b)$) the global optimal solution set (value) of (4),
- $S(a,b) = \{(x,y) \mid Qx + A^\top y = -a,\ 0 \le y \perp (Ax - b) \le 0\}$.

QP theory says: Defining

$$\Psi_{KKT}(a,b) = \operatorname{argmin}_{(x,y)}\{\tfrac{1}{2}(a^\top x - b^\top y) \mid (x,y) \in S(a,b)\}, \tag{5}$$

and its associated value function φ_{KKT}, one has

$$\boxed{\varphi(a,b) = \varphi_{KKT}(a,b) \quad \text{and} \quad \Psi(a,b) = \text{Proj}_{\mathbb{R}^n}\Psi_{KKT}(a,b)}$$

for all $(a,b) \in \text{dom}\,\Psi (\subset \text{dom}\,\Psi_{KKT})$.

As direct consequence of Theorem 1 (Robinson '76, '81), one has

$\boxed{S \text{ is polyhedral and hence satisfies the upper Lip. prop. (2).}}$

Moreover, there holds

Theorem 2. (Kl '85, see also Kl '87 (Proceedings Paraopt I))

1. $\operatorname{dom}\Psi$ is a finite union of polyhedral convex cones,

2. in general, the multifunction Ψ is not polyhedral (counterexample),

3. Ψ satisfies the upper Lip. prop. (2), and φ is Lipschitz on bounded subsets of $\operatorname{dom}\Psi$.

Assertion 3. recovers

Robinson '81 who assumed that Q is positive semidefinite, and

Kummer '77 who proved: φ continuous and Ψ (Hausdorff-) upper semi-continuous on $\operatorname{dom}\Psi$).

For positive (semi-)definite Q, refinements are possible: Guddat '76 and Bank et al. '82 extend Nožička's idea of local stability sets and obtain

Corollary 1. Suppose Q is positive semidefinite. Then

(i) Ψ is polyhedral, and $\operatorname{dom}\Psi$ is a convex polyhedral cone.

(ii) The value φ is continuous and piecewise quadratic on $\operatorname{dom}\Psi$.

(iii) If Q is positive definite, then the optimal solution function $\widehat{x}$ is Lipschitz and piecewise-affine on its domain.

Proof of (i):
by classical theory of convex quadratic programming.

Proof of (ii) and (iii) via **local stability sets**:

Given $I, J \subset \{1, \ldots, m\}$, define $(a,b) \in \Sigma^{I,J}$ iff relint$S(a,b)$ = set of all (x,y) such that

$$Qx + A^\top y = -a,\ (Ax)_i = b_i,\ i \in I,\ \ y_j = 0,\ j \in J, \\ (Ax)_i < b_i,\ i \notin I,\ \ y_j > 0,\ j \notin J. \tag{6}$$

(ii) With $(Q^{I,J})^+$ = pseudo-inverse of the matrix of equations in (6),

$$\begin{pmatrix} \widehat{x}(a,b) \\ \widehat{y}(a,b) \end{pmatrix} = (Q^{I,J})^+(-a, b_I, 0_J) \quad \text{for } (a,b) \in \Sigma^{I,J}, \tag{7}$$

defines an element $\widehat{x}(a,b)$ of the affine hull of $\Psi(a,b)$.

Hence,

$$\varphi(a,b) = \frac{1}{2}(\widehat{x}(a,b))^\top Q\,\widehat{x}(a,b) + a^\top \widehat{x}(a,b)$$

on cl $\Sigma^{I,J}$.

(iii) Theorem 2, Remark 1 and (7) $\Rightarrow$ $\widehat{x}$ Lipschitz and piecewise affine on dom Ψ.

Positive semidefinite Q, a fixed, b varies

Corollary 2. (Kl '84, recovered in Kl, Thiere '95)
Let Q be positive semidefinite, fix $a = \bar{a}$, and consider the QP

$$\min \ \frac{1}{2}x^\top Qx + \bar{a}^\top x \ \text{ s.t. } \ Ax \le b, \qquad b \text{ varies.}$$

Then the optimal set map $\widehat{\Psi} = \Psi(\bar{a}, \cdot)$ is Lipschitzian on $\operatorname{dom}\widehat{\Psi}$ w.r. to the Pompeiu-Hausdorff metric.

Sketch of proof: Combine the global (!) constants ϱ_Φ, ϱ_Ψ in

- $\forall b \, \exists \varepsilon > 0 : \widehat{\Psi}(b') \subset \widehat{\Psi}(b) + \varrho_\Psi \|b' - b\| B \ \forall b' \in B(b, \varepsilon)$, by Theorem 2,
- $\Phi(b, c) = \{x \mid Ax \le b, \ Qx = Qc, \ \bar{a}^\top x = \bar{a}^\top c\}$ is graph-convex and polyhedral, so Φ is Lipschitz on $\operatorname{dom}\Phi$ with constant $\varrho_\Phi > 0$,

use $\widehat{\Psi}(b) = \Phi(b, z_b, z_b) \ \forall z_b \in \widehat{\Psi}(b)$ to show

$\widehat{\Psi}$ is "pointwise Lipschitz" w.r. to d_H with global $\varrho = \varrho(\varrho_\Psi, \varrho_\Phi)$.

Since $\operatorname{dom}\widehat{\Psi}$ is convex, $\widehat{\Psi}$ is Lipschitzian on $\operatorname{dom}\widehat{\Psi}$, by Remark 1.

1. Upper Lipschitz stability in quadratic optimization

2. Further Lipschitz stability concepts and basic models

For brevity, KK '02 will refer to the book
Klatte, Kummer, *Nonsmooth Equations in Optimization*, Kluwer 2002.

Definition. Let $\Gamma : P \rightrightarrows Z$ (P, Z Banach spaces) be a multifunction and $\bar{z} \in \Gamma(\bar{p})$, B closed unit ball, $B(x, \varepsilon) := \{x\} + \varepsilon B$ (Minkowski sum).

Γ has the **Aubin property (Aub. p.)** at $(\bar{p}, \bar{z})$ if there are $\varepsilon, \delta, L > 0$,

$$\Gamma(p) \cap B(\bar{z}, \varepsilon) \subset \Gamma(p') + L\|p' - p\|B \quad \forall p, p' \in B(\bar{p}, \delta). \tag{8}$$

Note: The definition includes $\Gamma(p) \cap B(\bar{z}, \varepsilon) \neq \emptyset$ for p near $\bar{p}$.

Specializations

(i) **Calmness:** If (8) only holds for $p' = \bar{p}$, Γ is called **calm** at $(\bar{p}, \bar{z})$. However, $\Gamma(p) \cap B(\bar{z}, \varepsilon) = \emptyset$ for some p near $\bar{p}$ is possible.

(ii) **Strong Lipschitz stability:** If $\Gamma(p) \cap B(\bar{z}, \varepsilon) = \{z(p)\}$ for p near $\bar{p}$ in (8), Γ is called **locally single-valued and Lipschitz (l.s.L.)** at $(\bar{p}, \bar{z})$.

Relations to certain **regularity concepts:**

Γ has Aub.p. at $(\bar{p}, \bar{z}) \;\Leftrightarrow\; \Gamma^{-1}$ is metrically regular at $(\bar{z}, \bar{p})$,

Γ calm at $(\bar{p}, \bar{z}) \;\Leftrightarrow\; \Gamma^{-1}$ is metrically subregular at $(\bar{z}, \bar{p})$,

Γ l.s.L. at $(\bar{p}, \bar{z}) \;\Leftrightarrow\; \Gamma^{-1}$ is strongly (metrically) regular at $(\bar{z}, \bar{p})$.

Of course, for $P = \mathbb{R}^s$, $Z = \mathbb{R}^r$, calmness of Γ is a localized variant of the upper Lipschitz property (2), and one immediately has

If Γ satisfies (2), then Γ is calm at each $(p, z) \in \operatorname{gph} \Gamma$.

Basic models for the remaining paper

We study two models **with canonical perturbations** $p = (a, b)$ near $\bar{p} = 0$ and **convex polyhedral constraints** of the form

$$M(b) = \{x \in \mathbb{R}^n \mid Ax - c \le b\},$$

where $c \in \mathbb{R}^m$, A (m, n)-matrix (with rows A_i) are fixed.

Model 1. With $h \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, consider the **variational condition**

$$\text{VC}(a, b): \quad a \in \ h(x) + N_{M(b)}(x), \ x \in M(b).$$

Model 2. With $f \in C^2(\mathbb{R}^n, \mathbb{R})$, consider the **nonlinear program**

$$\text{NLP}(a, b): \qquad \text{Minimize } f(x) - \langle a, x \rangle \ \text{ s.t. } \ x \in M(b).$$

The **normal cone map** $(b, x) \in \operatorname{gph} M \mapsto N_{M(b)}(x)$ is polyhedral:

$$\begin{aligned} u \in N_{M(b)}(x) \quad &\Leftrightarrow \quad u^\top(\xi - x) \le 0 \;\; \forall \xi \in M(b) \\ &\Leftrightarrow \quad x \in \operatorname{argmax}_{\xi}\{u^\top \xi \mid A\xi - c \le b\} \\ &\Leftrightarrow \quad \exists \lambda \ge 0 : \;\; A^\top \lambda = u, \quad \lambda \perp Ax - c - b \le 0 \\ &\Leftrightarrow \quad \exists y \in \mathbb{R}^m : \;\; A^\top y^+ = u, \quad Ax - c - y^- = b \end{aligned}$$

Thus, $\boxed{x \text{ solves } \mathrm{VC}(a, b)}$ if and only if for some $y \in \mathbb{R}^m$,

$$\begin{array}{rcrcrcl} F_1(x,y) & := & h(x) & + & A^\top y^+ & = & a \\ F_2(x,y) & := & Ax - c & - & y^- & = & b \end{array} \qquad \textbf{(Kojima's form)}, \qquad (9)$$

or, equivalently, if and only if for some $\lambda \ge 0$

$$\begin{pmatrix} a \\ -b \end{pmatrix} \in \begin{pmatrix} h(x) + A^\top \lambda \\ c - Ax \end{pmatrix} + N_{\mathbb{R}^n \times \mathbb{R}^m_+}(x, \lambda). \qquad \textbf{(normal cone form)} \qquad (10)$$

With $\boxed{h = Df,}$ (9), or equivalently (10), describe the parametric KKT system of a solution x to NLP(a, b).

The correspondence $\lambda \leftrightarrow y$ between multipliers in (9) and (10) is a Lipschitzian homeomorphism, we prefer here the form (9).

Given any $p = (a, b)$, define for both model 1 and model 2

$$\begin{array}{rcll} S(p) & = & \{(x, y) \mid F(x, y) = p\}, & \textbf{(KKT solution set)} \\ X(p) & = & \{x \,|\, \exists y : F(x, y) = p\}, & \textbf{(stationary solution set)} \\ Y(p, x) & = & \{y \,|\, F(x, y) = p\}, & \textbf{(multiplier set w.r. to (9))} \\ \Lambda(p, x) & = & \{\lambda \,|\, \exists x : \ (x, \lambda) \text{ satisfies (10)}\}. & \textbf{(multiplier set w.r. to (10))} \end{array}$$

For some given solution $\bar{x} \in X(0)$ of the initial problem at $\bar{p} = 0$, let

$$Q := Dh(\bar{x}) \qquad (\text{or} := D^2 f(\bar{x})).$$

Example: Parametric linear programs - Lipschitz behavior

$$\min \quad -\sum_{i=1}^{2}(1+a_i)x_i$$

$$\text{s.t.} \quad x_1 - \tfrac{1}{2} \le b_1$$

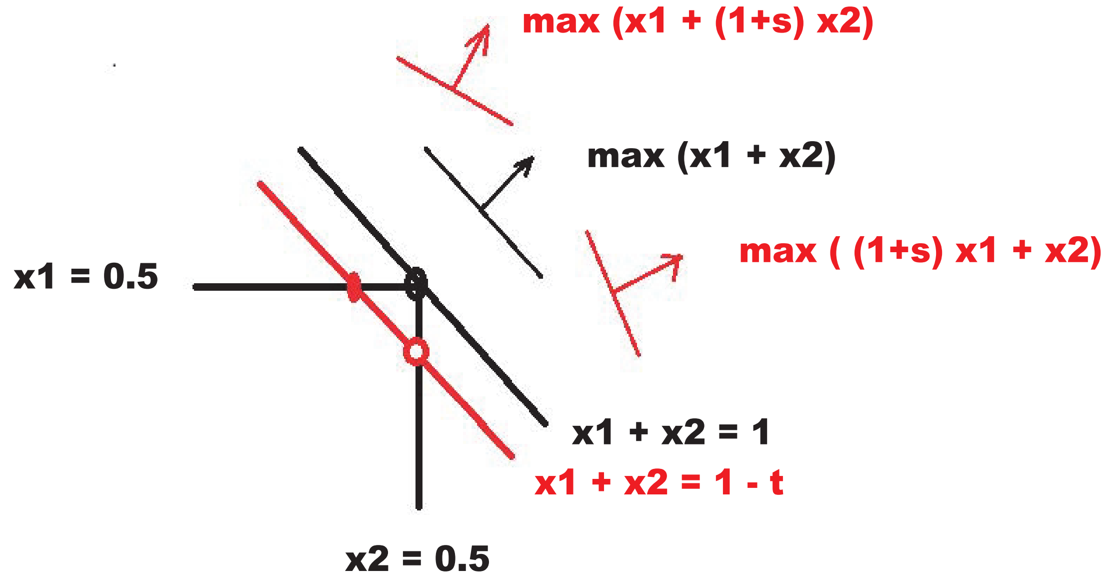
$$x_2 - \tfrac{1}{2} \le b_2$$

$$x_1 + x_2 - 1 \le b_3$$

$(a,b) \to (0,0)$

Argin mapping Ψ

is **(isolated) calm**



Has Ψ the Aubin property at the origin? **No!**

Proposition. (KK '02) Given any $f : Z \times P \to \mathbb{R}$ and $\mathcal{M} : P \rightrightarrows Z$ (Z Hilbert space), $\widetilde{\Psi}(p,a) =$ Argmin $\{f(z,p) - \langle a, z\rangle \mid z \in \mathcal{M}(p)\}$ has the Aubin property at $(\bar{p}, 0, \bar{z})$ only if $\widetilde{\Psi}$ is single-valued around $(\bar{p}, 0)$.

1. Upper Lipschitz stability in quadratic optimization

2. Further Lipschitz stability concepts and basic models

3. Lipschitz stability under constraint non-degeneracy

Let S and X be the maps associated with VC(a, b).

Theorem 3. If S has the Aubin property at $(0, (\bar{x}, \bar{y})) \in \operatorname{gph} S$ then

(i) LICQ holds at $\bar{x}$, i.e., $\{A_i \mid i : A_i\bar{x} = c_i\}$ is linearly independent,
(ii) S is locally single-valued and Lipschitz (l.s.L) at $(0, (\bar{x}, \bar{y}))$.

If h or Df are only locally Lipschitz, (i) holds, <u>but (ii) fails.</u>

Notes.

- Direct proof of (i),(ii) via structure of (9): Kummer '98, KK '02.
- Knowing (ii), (i) also follows from characterizations of l.s.L.
- Original proof of Theorem 3 (ii): by Dontchev, Rockafellar '96 - a consequence of its version on VI with **fixed** constraints

$$(*) \qquad p \in h(x) + N_K(x) \quad (p \text{ varies near } 0),$$

with solution set $\mathcal{X}(p)$, where $h \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, K convex polyhedron,

since the KKT system (10) of our $VC(a, b)$ is of type $(*)$.

Now linearize $(*)$ and consider at $\bar{x} \in \mathcal{X}(0)$

$$\text{the affine VI} \quad p \in h(\bar{x}) + Dh(\bar{x})(x - \bar{x}) + N_K(x),$$

with solution set $\mathcal{L}(p)$. Then, at $(0, \bar{x})$, and with AP = Aubin property:

$$\mathcal{X} \text{ AP} \Leftrightarrow \mathcal{L} \text{ AP} \Leftrightarrow \text{(strR): } \mathcal{L} \text{ l.s.L.} \Leftrightarrow \mathcal{X} \text{ l.s.L.}$$

Note: (strR) is Robinson's '80 strong regularity for system $(*)$.

Main tools in the original proof:

- Coderivative characterization for the AP, cf. Mordukhovich '93.
- Reduction theorem by Robinson '84, cf. also Robinson '91, '16:

  $\forall (x,u) \in \operatorname{gph} N_K : u+v \in N_K(x+w) \;\Leftrightarrow\; v \in N_{K_0}(w)$ if (v,w) small,

  where $K_0 := K_0(x,u) = \{w \in T_K(x) \mid w \perp u\}$ (critical cone),
- Theory of piecewise affine maps (openness, coherent orientation, one-to-one maps), cf. Scholtes '94, Robinson '92, Ralph '93.

Recent proofs: cf. Dontchev, Rockafellar '14, Ioffe '16.

Corollary 3. (Consequences for the stationary solution map X)

If X has the Aubin property at $(0, \bar{x})$, and LICQ holds at $\bar{x}$ w.r. to $Ax - c \leq 0$, then X is l.s.L. at $(0, \bar{x})$.

Idea of proof:

- Trivially, LICQ persists under small perturbations.
- Well-known consequence: LICQ implies that $\Lambda(p, x)$, $p = (a, b)$, is a singleton for $(p, x) \in \operatorname{gph} X$ near $(0, \bar{x})$, say $\Lambda(p, x) = \{\lambda(p, x)\}$. Moreover (cf. Kl, Tammer '90), $\lambda(\cdot, \cdot)$ has a representation as a polynomial in a, b, x and $h(x)$.
- So, for some neighborhood U of $(0, \bar{x})$, the multiplier map Λ is single-valued and Lipschitzian on $(\operatorname{dom} X \times \mathbb{R}^n) \cap U$.
- Standard estimation gives: X is Aubin $\Rightarrow$ S is Aubin.
- Apply Theorem 3.

Let $(\bar{x}, \bar{y}) \in S(0,0)$, $J = \{i \mid A_i\bar{x} = c_i\}$, $Q = Dh(\bar{x})$, A_i i-th row of A.

Characterization of "S is l.s.L." via complementarity

S is l.s.L at $(0, 0, \bar{x}, \bar{y})$ if and only if the **stability system**

$$\begin{array}{rlcll} (i) & Qu + \sum_{i \in J} \alpha_i A_i^\top & = & 0, & \\ (ii) & \bar{y}_i\, A_i\, u & = & 0 & (\forall i \in J) \\ (iii) & \alpha_i\ A_i\, u & \geq & 0 & (\forall i \in J) \end{array} \tag{11}$$

has only the solution $(u, \alpha_J) = (0, 0)$.

(11) includes: LICQ holds at $\bar{x}$, put $u = 0$.

Characterization of "S is l.s.L." via matrix criterion

Let $I^+ = \{i \mid \bar{y}_i > 0\}$, $I^0 = \{i \mid \bar{y}_i = 0\}$, and A_I matrix with rows A_i, $i \in I$, and $\widetilde{A}_I = [A_I\ 0]$. Then S is l.s.L at $(0, 0, \bar{x}, \bar{y})$ if and only if

$$R = \begin{pmatrix} Q & A_{I^+} \\ -A_{I^+} & 0 \end{pmatrix} \text{ nonsingular and } \widetilde{A}_{I^0}\, R^{-1}\, (\widetilde{A}_{I^0})^\top \text{ is a P-matrix.} \tag{12}$$

Lipschitz stability for general models/perturbations

Both criteria were in fact proved as characterizations of l.s.L. for the solution map $S(a,b,t)$ of the following general model (similarly (10)'):
Let $h \in C^1$, $g \in C^2$ and consider parameter (a,b,t) near $(0,0,\bar{t})$,

$$\begin{array}{rcllcl} F_1(x,y) & := & h(x,t) & + & \sum_{i=1}^m y_i^+ D_x g_i(x,t)^\mathsf{T} & = & a \\ F_2(x,y) & := & g(x,t) & - & y^- & = & b \end{array} \qquad (9)'$$

put in (11), (12) $Q = D_x h(\bar{x},\bar{t}) + \sum \bar{y}_i^+ D_x^2 g_i(\bar{x},\bar{t})$, $A_i = D_x g_i(\bar{x},\bar{t})$.

Condition (11): first given by Kummer '91 as a characterization via injectivity of the strict graphical (Thibault) derivative, see also KK '02, Facchinei, Pang '03, Dontchev, Rockafellar '14.

Condition (12): goes back on classical work by Robinson '80, nice straightforward proof in Outrata, Kočvara, Zowe '98, for equivalent matrix criteria cf. also Kojima '80, Jongen et al. '87, Bonnans, Shapiro '00, KK '02, Facchinei, Pang '03, Dontchev, Rockafellar '14.

Extension of Thm. 3 and Cor. 3 to non-polyhedral constraints

Let $(\bar{x}, \bar{\lambda}) \in \mathbb{R}^n \times \mathbb{R}^m$ be a solution of the equation $0 \in \mathcal{F}(x, \lambda)$, where $\mathcal{F}$ is the KKT multifunction associated with

$$P(a,b): \qquad \begin{array}{ll} \min_x & f(x) - \langle a, x \rangle \\ \text{subject to} & g(x) - b \in K, \end{array}$$

Suppose K closed convex cone, and here $f, g \in C^1$.

Let $S(a,b) = \mathcal{F}^{-1}(a,b)$, (a,b) near 0, and

$$\Lambda(x, a, b) = \{\lambda \,|\, (x, \lambda) \in S(a,b)\}\,.$$

Theorem 3(i)'. (Kl, Kummer '13)

Given a KKT point $(\bar{x}, \bar{\lambda}) \in S(0)$, suppose S has the Aubin property at $(0, \bar{x}, \bar{\lambda})$. Then $g(x) \in K$ is nondegenerate at $\bar{x}$, i.e.,

$$\Big[\ (*) \quad Dg(\bar{x})^{\mathsf{T}} u = 0 \text{ and } u \in \operatorname{span} N_K(g(\bar{x}))\ \Big] \quad \text{imply } u = 0.$$

and so $\Lambda(\bar{x}, 0) = \{\bar{\lambda}\}$.

"metric = strong metric regularity" for non-polyhedral K:

Optimal solution sets for convex optimization problems

Convex problems: f is convex and the graph of $G(x) = g(x) + K$ is convex, $f, g \in C^1$, cf. Kl, Kummer '13.

Stationary solution sets of local minimizers

SOCP: $K =$ 2nd-order cone, $f, g \in C^2$, cf. Outrata, Ramirez '11 including algebraic characterizations.
SDP: $K = S^m_+$, $f, g \in C^2$, cf. Fusek 13. Characterizations for "X l.s.L." are known for SDPs e.g. from Bonnans, Shapiro '00, D.Sun '06, Chan, D.Sun '08.

Note. The mentioned characterizations of l.s.L. for SOCP/SDP include non-degeneracy.

1. Upper Lipschitz stability in quadratic optimization

2. Further Lipschitz stability concepts and basic models

3. Lipschitz stability under constraint non-degeneracy

4. **Lipschitz stability for degenerate constraints**

Let X be the stationary solution map X of VC(a, b), $(0, \bar{x}) \in \operatorname{gph} X$ and $J = \{i \mid A_i\bar{x} = c_i\}$.

We say that **the stability system (11),**

$$
\begin{array}{rlcll}
(i) & Qu + \sum_{i \in J} \alpha_i A_i^\top & = & 0, & \\
(ii) & y_i\, A_i\, u & = & 0 & (\forall i \in J), \\
(iii) & \alpha_i \;\, A_i\, u & \geq & 0 & (\forall i \in J),
\end{array}
\qquad (11)
$$

is nonsingular at (0,x̄),

if for each $y \in Y(0, \bar{x})$, (11) has only solutions (u, α_J) with $u = 0$.

Theorem 4. Let $(0, \bar{x}) \in \operatorname{gph} X$.

(i) If X has the Aubin property at $(0, \bar{x})$ then MFCQ holds at $\bar{x}$, i.e. $\exists u : \; A_i u < 0 \; \forall i \in J$ (= Slater's CQ in our model).

(ii) X is l.s.L. at $(0, \bar{x})$ if and only if both $\bar{x}$ satisfies MFCQ and the stability system (11) is nonsingular at $(0, \bar{x})$.

∘ (i) and (ii)"only if" also hold for $h \in C^{0,1}$, $Qu := Th(\bar{x}; u)$ (KK '02).
∘ Linearity of constraints is only essential for "(ii) if-direction", cf. Kl, Kummer '05.
∘ Aubin = l.s.L.? open to my knowledge

L.s.L. and strong coherent orientation condition (SCOC)

Under MFCQ at $\bar{x}$, the standard multiplier set $\Lambda^0 = \Lambda(0, \bar{x})$ is a bounded polyhedron, let Λ^e be the set of its vertices.

Let $J = \{i \mid A_i\bar{x} = c_i\}$. Given $\lambda \in \Lambda^e$, put $I^+(\lambda) = \{i \mid \lambda_i > 0\}$ and

$$\mathcal{T}(\lambda) = \{I \mid I^+(\lambda) \subset I \subset J,\ A_I \text{ has full row rank}\}.$$

Corollary 4 X is l.s.L. at $(0, \bar{x})$ if and only if both $\bar{x}$ satisfies MFCQ and (SCOC) Luo et al. '96 is satisfied at $(0, \bar{x})$, i.e.,

For all $\lambda \in \Lambda^e$, the determinants of the matrices

$$R(\lambda) = \begin{pmatrix} Q & A_{I+(\lambda)} \\ -A_{I+(\lambda)} & 0 \end{pmatrix} \text{ have the same sign } \pm 1, \tag{13}$$

and for each $I \in \mathcal{T}(\lambda)$, $\widetilde{A}_I R(\lambda)^{-1} (\widetilde{A})_I^\top$ is a P-matrix.

Note. Lu, Robinson '08 give a more general version of Corollary 4, allowing in VC(a, b) only perturbations $b \in \operatorname{dom} M$ to avoid MFCQ, where $M(b)$ constraint set map.

What about nonlinear constraints or perturbations?

$\exists$ counterexamples showing Theorem 4 (ii) cannot be carried over to

1° perturbed constraints of type $A(t)x - c \leq b$, see Robinson '82,

2° perturbed constraints $g(x) \leq b$, g_i convex polynomials, one needs some condition involving limits $x^k \to \bar{x}$, cf. Kl, Kummer '05.

Counterexample for 2°. See next slide.

Counterexample for 2°. (Kl, Kummer '05)

$$\min\ \tfrac{1}{2}x_1^2 + x_2 - a_1 x_1 - a_2 x_2 \quad \text{s.t.} \quad -x_2 \le b_1,\ x_1^2 - x_2 \le b_2.$$

Then $\bar{x} = (0,0)$ is a minimizer at $(a,b) = (0,0)$, MFCQ holds. Let

$$\begin{aligned} W^+(\bar{x},\lambda) &= \{u \mid u^\top D g_i(\bar{x}) = 0,\ i \in I^+(\lambda)\}, \\ Q(\lambda) &= D^2 f(\bar{x}) + \textstyle\sum \lambda_i D^2 g_i(\bar{x}). \end{aligned}$$

Obviously, $\Lambda(\bar{x}) = \{\lambda \ge 0 \mid \lambda_1 + \lambda_2 = 1\}$ and

$$Q(\lambda) = \begin{pmatrix} 1 + 2\lambda_1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\lambda \in \Lambda(\bar{x}) \Rightarrow \substack{\lambda_1 > 0 \\ or\ \lambda_2 > 0} \Rightarrow W^+(\bar{x},\lambda) = \mathbb{R} \times \{0\}.$$

Thus, $Q(\lambda)$ is positive definite on $W^+(\bar{x},\lambda)$ $(\forall \lambda \in \Lambda(\bar{x}))$.

Hence, the strong 2nd-order sufficient optimality condition (strong SSOC) plus MFCQ hold true, **but one can show:**

the stationary/optimal solution map X is <u>not l.s.L.</u>

In contrast to our example

$$\min \ \frac{1}{2}x_1^2 + x_2 - a_1x_1 - a_2x_2 \quad \text{s.t.} \quad -x_2 \le b_1,\ x_1^2 - x_2 \le b_2,$$

the problem with linearized constraints

$$\min \ \frac{1}{2}x_1^2 + x_2 - a_1x_1 - a_2x_2 \ \text{ s.t. } \ -x_2 \le b_1, -x_2 \le b_2,$$

has a l.s.L. solution map X near the origin and MFCQ plus strong SSOSC hold (special case of Corollary 4 for local minimizers).

Consequently,

$\boxed{\text{characterizations for l.s.L. of } X \text{ and that for linearized } g \text{ differ!}}$

Ways out:

- Require in addition Constant Rank CQ, see Luo et al.'96, Facchinei, Pang'03, or study weaker types of Lipschitz stability.
- Study in 2° "l.s. and Hölder" under MFCQ, as e.g. Gfrerer' 87.

Note. For linear SIP, the solution set mapping X is l.s.L. (= Aubin) iff Slater CQ plus some strong uniqueness condition are satisfied, cf. Canovas et al. '07.

1. Upper Lipschitz stability in quadratic optimization

2. Further Lipschitz stability concepts and basic models

3. Lipschitz stability under constraint non-degeneracy

4. Lipschitz stability for degenerate constraints

5. **Concluding remarks**

Many important and well-studied approaches and results in our context of variational problems involving polyhedral multifunctions were omitted or only sketched in this talk:

- extensions to the case of locally Lipschitz h or Df,
- study of optimization problems min $f(p, x)$ s.t. $g(p, x) \in K$, K finite union of convex polyhedra, e.g. perturbed MPECs,
- characterizations of calmness or isolated calmness of S or X,
- characterizations of l.s.L. if $\bar{x}$ is a local minimizer in NLP,
- study of the related modulus or radius of Lipschitz stability,
- analysis under special perturbations, where, e.g., metric and strong regularity differ in general, or directional CQs could be used,
- applications to convergence analysis of algorithms.

4 * main contributors to the area of this talk have a jubilee in 2017:

Stephen M. Robinson celebrated his 75th birthday.

Bert Jongen celebrated his 70th birthday.

Bernd Kummer celebrated his 70th birthday.

Jiři Outrata celebrated his 70th birthday.

Congratulation!

* Thanks to Jan Rückmann for extending the original list which might be even longer

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