The identification of beliefs from asset demand

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Abstract

The demand for assets as prices and initial wealth vary identifies beliefs and attitudes towards risk. We derive conditions that guarantee identification with no knowledge either of the cardinal utility index (attitudes towards risk) or of the distribution of future endowments or payoffs of assets; the argument applies even if the asset market is incomplete and demand is observed only locally.

Key words: identification; beliefs; attitudes towards risk; asset prices.

JEL classification: D80; G10.

1 Introduction

We consider an individual who trades in financial assets to maximize his stationary and time-separable (subjective) expected utility over two dates; and we assume that we observe how his initial date demand for consumption and assets varies with prices and wealth, while the investor's beliefs over stochastic asset payoffs and endowments, that are unobservable, remain fixed. We investigate conditions under which one can identify the investor's beliefs and attitudes towards risk, the cardinal utility index, from his demand for assets.

It is clear that beliefs cannot always be identified even when the support of risky endowments and asset payoffs is known. For example, if the investor has quadratic utility and no endowments in the second period, his demand for assets only depends on the first and the second moment of asset payoffs – higher moments are irrelevant, and, as a consequence, beliefs about higher moments of the distribution cannot possibly be identified.

More interestingly, if the investor has log-utility, the entire distribution matters for his utility; however, if the investor does not have (labor) endowments beyond the initial date, and if there is a single, risky asset available for trade, the demand for this asset only depends on the wealth of the investor and his discount factor; beliefs over the payoffs of the asset do not matter and cannot be identified. When the support of stochastic endowments is not known, the identification of beliefs may not be possible even if financial markets are complete. Since the observation of the demand for assets is equivalent to the observation of *excess demand* (but not necessarily consumption), the identification of beliefs turns out to be impossible if the cardinal utility exhibits constant absolute risk aversion.

We derive conditions on fundamentals that ensure that beliefs can be identified. Cardinal utility can always be identified (locally) from demand for date 0 consumption. Our main result is that identification is possible if the indirect marginal utilities for assets, across realizations of uncertainty, are linearly independent. We show that this condition is satisfied in a wide variety of situations. Without any assumptions on the risky assets, in the presence of a risk-free asset, beliefs over endowments and payoffs of assets can be identified if, for any $K > 1$ and any distinct $\{e_k\}_{k=1}^K$, the functions $\{u'(e_k + x)\}_{k=1}^K$ are linearly independent. We characterize classes of utility functions with this property, and we provide an argument for generic identification. If the payoffs of risky assets *separate uncertainty* in the sense that, for any two states, some portfolio of assets has different payoffs across the two states, beliefs can be identified if the individual's endowments are known (for example, because they are 0 across all states) or if no derivative of cardinal utility is the product of the exponential function and some periodic function.

Moreover, we show that, if cardinal utility is analytic, in the presence of a risky asset that separates uncertainty, one can drop the assumption that utility is stationary: identification is possible even when first and second period cardinal utilities can be different. And we give an extension of the argument that tackles models with more than two dates.

In conclusion, we argue that the analysis extends to the case in which only aggregate demand or the equilibrium correspondence are observable.

The identification of fundamentals is of intrinsic theoretical interest; also, it serves to formulate policy: here, imperfections, like market incompleteness, are of interest, since these imperfections render interventions desirable. And, it is essential in order to understand better paradoxes that arise in classical consumption based asset pricing. In financial markets, prices are thought to be determined by the joint probability distribution of payoffs and idiosyncratic shocks to investors, as well as their risk-preferences. Deviations of the prices of assets from these "fundamentals" are often attributed to the beliefs of investors. Perhaps most famously in Shiller (2015), unusual run-ups in asset prices are described as "irrational exuberance". In order to investigate the extent to which asset prices are determined by fundamentals or the beliefs of investors, it is necessary to identify these beliefs from market data. It is an open question to what extent this is possible in general. We investigate this question under the strong assumption that demand is observable, but we make few assumptions on the beliefs of the investor over asset payoffs and his endowments or the structure of the asset market.

One might wonder under which conditions the assumption of observable demand is justified in practice. Our methods can potentially be applied to data obtained from laboratory experiments, as, for example, in Choi, Fishman, Gale, and Kariv (2007) or Asparouhova, Bossaerts, Eguia, and Zame (2015); or with modifications to market data obtain from auctions, as in Hortaçsu and Kastl (2012). In all of these cases one obtains a finite number of observations on prices, incomes and individual demands. With a finite set of observations, Varian (1983) provided conditions necessary and sufficient for portfolio choices to be generated by expected utility maximization with a known distribution of payoffs; which extends the characterization of Afriat (1967). For the case of complete financial markets, Kübler, Selden, and Wei (2014) refined the argument to eliminate quantifiers and obtain an operational characterization. In the same vane, Echenique and Saito (2015) extended the argument to the case of subjective expected utility where beliefs are unknown. As an extension to our main results, we show that identification guarantees the convergence of preferences and beliefs constructed in Varian (1983) or Echenique and Saito (2015) to a unique profile as the number of observations becomes dense.

The identification of fundamentals from observable data can be addressed, most simply, in the context of certainty; there Mas-Colell (1977) showed that the demand function identifies the preferences of the consumer, while Chiappori, Ekeland, Kubler, and Polemarchakis (2004) extended the argument to show that aggregate demand or the equilibrium correspondence, as endowments vary, also allow for identification. Importantly, the argument for identification is local: if prices, in the case of demand, or endowments, in the case of equilibrium, are restricted to an open neighborhood, they identify fundamentals in an associated neighborhood. Evidently, the arguments extend to economies under uncertainty, but with a complete system of markets in elementary securities.

Identification becomes problematic, and more interesting, when the set of observations is restricted. Under uncertainty, this arises when the asset market is incomplete and the payoffs to investors are restricted to a subspace of possible payoffs. Nevertheless, Green, Lau, and Polemarchakis (1979), Dybvig and Polemarchakis (1981) and Geanakoplos and Polemarchakis (1990) demonstrated that identification is possible as long as the utility function has an expected utility representation with a state-independent cardinal utility index, and the distribution of asset payoffs is known. Polemarchakis (1983) extended the argument to the joint identification of tastes and beliefs; but, the argument relied crucially on the presence of a risk-free asset and, more importantly, did not allow uncertainty due to future endowments.

It is interesting to note that the identification of preferences from the excess demand for commodities, that corresponds to the demand for elementary securities in a complete asset market, is, in general, not possible, as shown in Chiappori and Ekeland (2004) and Polemarchakis (1979). Here, restrictions on preferences, additive separability and stationarity or state-independence, allow for identification even in an asset market that is incomplete.

A strand of literature in finance, most recently Ross (2015) and earlier work by He and Leland (1993), Wang (1993), Dybvig and Rogers (1997), Cuoco and Zapatero (2000) and Carr and Yu (2012), focuses on supporting prices and observations for a single realization of the path of endowments or, equivalently, on equilibrium, in an economy with a representative investor. In particular, Ross (2015) provides a simple framework where beliefs can be identified from asset prices. However, to obtain the result he needs to assume that there is a single (representative) agent, markets are complete and, importantly, the economy is stationary in levels as Borovicka, Hansen, and Scheinkman (2016) point out. In models with heterogeneous agents and incomplete financial markets or heterogeneous beliefs across agents an individual's consumption will never be Markovian and therefore this approach cannot be extended to make any statements about individuals' beliefs.

2 Identification

Dates are $t = 0, 1$, and, at each date-event, there is a single perishable good. At date 0, assets, $a = 1, \ldots, A$, are traded and they pay off at $t = 1$.

An individual has subjective beliefs over the joint distribution of asset-payoffs and his endowments at $t = 1$ that, we assume, has finite support, $\{1, \ldots, S\}$.

Consumption is x_0 at date 0, and it is x_s at state of the world $s = 1, \ldots, S$ at date 1. The individual maximizes time-separable expected utility

$$U(x_0, \ldots, x_s, \ldots) = u(x_0) + \beta \sum_{s=1}^{S} \pi_s u(x_s),$$

with the cardinal utility index, $u : (0, \infty) \to \mathbb{R}$, continuously differentiable, strictly concave and strictly monotonically increasing, $\beta \in (0, \infty)$ and π a probability measure. Payoffs of an asset are $r_a = (r_{a,1}, \ldots, r_{a,s}, \ldots, r_{a,S})^\top$, and payoffs of assets at a state of the world are $R_s = (r_{1,s}, \ldots, r_{a,s}, \ldots, r_{A,s})$. Holdings of assets, portfolios, are $y = (\ldots, y_a, \ldots)^\top$. At date 0, the endowment of the individual is e_0, consumption is numéraire and prices of assets are $q = (\ldots, q_a, \ldots)$; at state of the world $s = 1, \ldots, S$, at date 1, consumption is, again, numéraire, and the endowment is e_s; across states of the world, $e = (e_1, \ldots, e_S)$.

The optimization problem of the individual is

$$\begin{array}{ll} \max_{x \geq 0, y} & u(x_0) + \beta \sum_{s=1}^{S} \pi_s u(x_s) \\ \text{s.t.} & x_0 + qy \leq e_0, \\ & x_s - R_s y \leq e_s, \ \ s = 1, ..., S. \end{array}$$

The demand function for consumption and assets is $(x_0, y)(q, e_0)$; it defines the inverse demand function $(q, e_0)(x_0, y)$. For a given $(\bar{q}, \bar{e}_0)$, we suppose that $(x_0, y)(q, e_0)$ is observable and solves the individual's maximization problem, with $(x_0, \ldots, x_s, \ldots) \gg 0$ on an open neighborhood of $(\bar{q}, \bar{e}_0)$ [1]. We assume that asset demand is continuous and invertible for the observed prices

[1] We assume, throughout, that, at observed prices and incomes, consumption is strictly positive. This simplifies the analysis and can of course be ensured by assuming an Inada condition on $u(\cdot)$. Many of our results extend to the case of consumption on the boundary.

and date 0 incomes. This implicitly imposes the restriction that the individual believes that the observed prices are arbitrage-free. We will make this assumption throughout the paper without stating it again. With this assumption the observed prices and incomes are associated with an open set of observed asset holdings and date 0 consumption. We denote by $\mathcal{Y} \subset \mathbb{R}^A$ the projection of this set on asset holdings and by $\mathcal{X}_0$ the projection on date 0 consumptions. Unobservable characteristics of an individual are the cardinal utility index, $u : (0, \infty) \to \mathbb{R}$, the discount factor, $\beta > 0$ and beliefs over the distribution of future endowments and payoffs of assets, $S \in \mathbb{N}$, $(\pi, R, e) \in \mathbb{R}^S_+ \times \mathbb{R}^{AS} \times \mathbb{R}^S_+$, with $\pi = (\ldots, \pi_s, \ldots)$ a probability measure. Does the demand function identify the unobservable characteristics of the individual? This is the question we address in this paper.

The following result establishes that the cardinal utility index can be identified over the range of observable date 0 consumption.

Lemma 1. *The demand function for consumption and assets identifies the cardinal utility index* $u : \mathcal{X}_0 \to \mathbb{R}$ *up to an affine transformation.*

Proof. The demand for consumption and assets is defined by the the first order conditions

$$\beta \sum_{s=1}^{S} \pi_s u'(e_s + R_s y) = u'(x_0) q(x_0, y).$$

Normalizing $u'(\bar{x}_0) = 1$, at some $\bar{x}_0 \in \mathcal{X}_0$, we obtain that

$$u'(x_0) q_a(x_0, y) - q_a(\bar{x}_0, y) = 0,$$

for any $a = 1, \ldots, A$, and for all $x_0 \in \mathcal{X}_0$, which identifies $u'(x_0)$, for $q_a(x_0, y) \neq 0$, since inverse demand is observable.

□

Remark 1. Note that if $u(\cdot)$ is assumed to be analytic on $(0, \infty)$, the observation of demand on any open $\mathcal{X}_0 \subset (0, \infty)$ identifies the cardinal utility on all of $(0, \infty)$; for this case, we take $\mathcal{X}_0$ to be equal to $(0, \infty)$ in the results that follow.

With $u(\cdot)$ given, the unknown characteristics are $\xi = (S, \beta, \pi, R, e)$, and the question of identification is whether, given some ξ that generates the observed demand function, there is a distinct $\tilde{\xi}$ that would generate the same demand for assets on the specified neighborhood of prices and wealth. While we do not provide a complete answer to this question, we give conditions on admissible characteristics that ensure identification. First, note that we must obviously assume that there is a portfolio of assets that has a positive payoff in all subsequent nodes. Probabilities of nodes at which no asset pays off cannot possibly be identified. Without loss of generality, we therefore assume that $r_{1s} > 0, s = 1, ..., S$. Moreover, two states for which endowments and asset pay-offs are identical are taken to be the same state; that is, we assume that there are no s, s' with $(e_s, R_s) = (e_{s'}, R_{s'})$. It is clear that beliefs just depend on the distribution of endowments and asset payoffs and not on the state in which the assets pay off. To avoid the ambiguity that this introduces we assume that $s < s'$ if $(e_s, R_s) < (e_{s'}, R_{s'})$ in the lexicographic order.

Identification is possible if any two distinct admissible characteristics generate different demand functions. Formally, we say that the observed demand for date 0 consumption and for assets (on an open set of incomes and prices) identifies beliefs (in a set of admissible characteristics Ξ) if there are no distinct $\xi^1, \xi^2 \in \Xi$ that, for the cardinal utility $u(\cdot)$ recovered in Lemma 1, generate the same demand function on the observed set of prices and incomes. Our main results states that ξ^1 and ξ^2 must generate different demand functions if $\{u'(e_s + R_s y)\}$ are linearly independent for all s for which (e_s, R_s) are distinct.

To state the theorem formally, define the $(n-1)$-dimensional unit sphere, $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i^2 = 1\}$. Recall that functions, $\{f_i\}_{i=1}^n$, with $f_i : \mathcal{A} \subset \mathbb{R}^m \to \mathbb{R}$, $i = 1, .., n$, are linearly independent on $\mathcal{A}$ if there is no $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{S}^{n-1}$, such that $\sum_{i=1}^n \alpha_i f_i(x) = 0$, for all $x \in \mathcal{A}$.

If functions $f_1, \ldots, f_n$ are linearly independent on some set $\mathcal{A}$ there must exist finitely many points $x_1, \ldots, x_m \in \mathcal{A}$ such that there is no $\alpha \in \mathbb{S}^{n-1}$ for which $\sum_{i=1}^n \alpha_i f(x_j) = 0$ for all $j = 1, ..., m$. To see this note that independence implies that for any $\bar{\alpha} \in \mathbb{S}^{n-1}$ there is some $\bar{x} \in \mathcal{A}$ such that $\sum_{i=1}^n \bar{\alpha}_i f(\bar{x}) \neq 0$. Since the function $\sum_{i=1}^n \alpha_i f(\bar{x})$ is continuous in α there must be some open neighborhood around $\bar{\alpha}$ such that $\sum_{i=1}^n \alpha_i f(\bar{x}) \neq 0$ for all α in that neighborhood. Compactness of $\mathbb{S}^{n-1}$ then implies the result.

Defining a differential operator

$$\Delta_k = (\frac{\partial}{\partial x_1})^{j_1} \ldots (\frac{\partial}{\partial x_m})^{j_m}, \quad j_1 + \ldots + j_m \leq k,$$

we say that $f_1, \ldots, f_n$ are *differentiably* linearly independent (on $\mathcal{A}$) if there is some $k \geq n-1$ and some $\bar{x} \in \mathcal{A}$, such that each f_i is at least C^k at $\bar{x}$ and such that there are differential operators $\Delta_{k_1}, ..., \Delta_{k_n}$, with $k_i \leq k$, for all $i = 1, \ldots, n$, such that the matrix

$$\widetilde{W} = \begin{pmatrix} \Delta_{k_1}(f_1) & \dots & \Delta_{k_1}(f_i) & \dots & \Delta_{k_1}(f_n) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Delta_{k_j}(f_1) & \dots & \Delta_{k_j}(f_i) & \dots & \Delta_{k_j}(f_n) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Delta_{k_n}(f_1) & \dots & \Delta_{k_n}(f_i) & \dots & \Delta_{k_n}(f_n) \end{pmatrix} \quad (1)$$

is non-singular.

It is easy to see that, if $f_1, \dots, f_n$ are differentiably linearly independent on $\mathcal{A}$, they are linearly independent since differentiable linear independence implies that there cannot be an open neighborhood of $\bar{x}$ and some $\alpha \in \mathbb{S}^{N-1}$ such that $\sum_{i=1}^n \alpha_i f(x_i) = 0$ for all x in the neighborhood.

The converse is generally not true. But, Bostan and Dumas (2010) show that, if the functions $f_1, ..., f_n$ are analytic, then, they are linearly independent if and only if they are differentiably linearly independent. In fact in this case one can take $\Delta_{k_i} = \Delta_{i-1}$ for all $i = 1, ..., n$ and obtains the so called Wronsikian matrix of $f_1, ..., f_n$. In the analytic case, this matrix is non-singular if and only if the functions are independent.

Given characteristics $\xi^1 = (S^1, \beta^1, \pi^1, R^1, e^1)$, $\xi^2 = (S^2, \beta^2, \pi^2, R^2, e^2) \in \Xi$, we define the joint support $(\mathcal{S}, \bar{e}, \bar{R}) = \Sigma(\xi^1, \xi^2)$ as $\bar{e}_s = e^1_s$, $\bar{R}_s = R^1_s$ for $s \leq S^1$ and $\bar{e}_s = e^2_{s-S^1}$, $\bar{R}_s = R^2_{s-S^1}$ for $s > S^1$ as well as $\mathcal{S} = \{1, \dots, S^1\} \cup \{s \in \{S^1 + 1, \dots, S^1 + S^2\} : (\bar{e}_s, \bar{R}_s) \notin \{(e_1, R_1), \dots, (e_{S^1}, R_{S^1})\}$. We also define the open set $\mathcal{Y}_{\bar{e},\bar{R}} = \{y \in \mathcal{Y} : (\bar{e}_s + y\bar{R}_s) \in \mathcal{X}_0, \text{ for all } s \in \mathcal{S}\}$ of portfolios that ensure that consumption at any state of the world, s, at date 1 lies in $\mathcal{X}_0$.

It is now possible to give general necessary and sufficient conditions for identification.

Theorem 1 (Identification)**.** *The demand function identifies the unobservable characteristics, $u(\cdot)$, up to an affine transformation, and $\xi \in \Xi$ if, for any $\xi^1, \xi^2 \in \Xi$, the joint support $(\mathcal{S}, \bar{e}, \bar{R}) = \Sigma(\xi^1, \xi^2)$ is such that the set $\mathcal{Y}_{\bar{e},\bar{R}}$ is non-empty, and the functions $\{u'(\bar{e}_s + \bar{R}_s y)\}_{s \in \mathcal{S}}$ are linearly independent on this set.*

Conversely, if there are characteristics $\xi = (S, \beta, \pi, R, e) \in \Xi$ for which $\{u(e_s + R_s y)\}_{s=1}^S$ are not linearly independent on $\mathcal{Y}$, then identification is impossible.

Proof. To prove sufficiency first note that Lemma 1 shows identification of the cardinal utility index, $u(\cdot)$. To show that $\xi \in \Xi$ is identified, suppose both characteristics ξ^1 and ξ^2 rationalize observed asset demand and consider asset demand in a fictitious problem of an investor who faces states $\mathcal{S}$ (as defined in the theorem) in the second date. Define $f_s(y) = u'(\bar{e}_s + \bar{R}_s y)$. The first order condition with respect to the demand for asset $a = 1$ (that has positive payoffs in all states) can be written as

$$\sum_{s \in \mathcal{S}} \beta \pi_s \bar{r}_{1s} f_s(y) = u'(x_0) q_1.$$

We first show that the fact that this first order condition holds on an open neighborhood uniquely determines $\beta, \{\pi_s\}_{s \in \mathcal{S}}$. We then argue that this implies that the demand function identifies beliefs.

As pointed out above, if the f_s are linearly independent, we can find a positive integer N and points $y_1, \ldots, y_N \in \mathcal{Y}_{\bar{e},\bar{R}}$ such that the system of equations

$$\sum_{s \in \mathcal{S}} \alpha_s \bar{r}_{1s} f_s(y_i) = 0, \quad i = 1, ..., N$$

has no solution with $\alpha \neq 0$. Since the first order conditions hold on the open set $\mathcal{Y}_{\bar{e},\bar{R}}$, we can find $\{(x_{0i}, q_{1i})\}_{i=1}^n$, such that

$$\sum_{s \in \mathcal{S}} \beta \pi_s \bar{r}_{1s} f_s(y_i) = u'(x_{0i}) q_{1i}, \quad i = 1, ..., N.$$

This is a linear system in $\{\beta \pi_s\}_{s \in \mathcal{S}}$, and it must have a unique solution. By the construction of the set of distinguishable states $\mathcal{S}$, if ξ^1 rationalizes the observed demand, this solution must satisfy $\beta \pi_s = 0$ for all $s > S^1$. But then, if ξ^2 also rationalizes the observed demand we must have $S^1 = S^2$ and $(e_s^1, R_s^1) = (e_s^2, R_s^2)$, for all $s = 1, ..., S^1$. Hence characteristics are uniquely identified.

To prove necessity, note that linear dependence of $\{u(e_s + R_s y)\}$ implies that there exist $\alpha_1, \ldots, \alpha_S$ such that $\sum_{s=1}^S \alpha_s r_{as} u'(e_s + R_s y) = 0$, for all $a = 1, ..., A$. If asset demand is rationalized for some probabilities $(\pi_1, \ldots, \pi_S) \gg 0$ then, for any ϵ, the first order conditions can be written as follows:

$$-q_a u'(x_0) + \sum_{s=1}^{S} (\pi_s + \epsilon \alpha_s) r_{as} u'(e_s + R_s y) = 0, \text{ for all } a = 1, \ldots, A.$$

For sufficiently small $\epsilon > 0$ we have that $\pi_s + \epsilon \alpha_s > 0$ for all s, and we can define alternative probabilities $\tilde{\pi}_s = (\pi_s + \epsilon \alpha_s)/(1 + \epsilon \sum_k \alpha_k)$ and appropriately adjusted $\tilde{\beta} = \beta(1 + \epsilon \sum_k \alpha_k)$ that would rationalize the same demand function.

□

The identification theorem obviously raises the question whether there are assumptions on fundamentals, either assumptions on utility or restrictions on (e_s, R_s) that guarantee independence as required. Before addressing this question in Section 3 below we first show that the Identification Theorem can be applied to a revealed preference framework.

2.1 Afriat inequalities and revealed beliefs

Our theoretical result about the possibility of identification has implications for revealed preference analysis. While with a finite number of observations beliefs obviously cannot be identified, we show that as the number of observations goes to infinity and the observations become dense the underlying characteristics can be uniquely recovered.

We assume that the number of states is fixed at some S, and one observes a collection of pairs $\{((x_0^i, y^i), (q^i, e_0^i))\}_{i=1}^I$, of choices of date 0 consumption and holdings of assets, $\{x_0^i, y^i\}$, at prices and incomes $\{q^i, e_0^i\}$. We assume throughout that $x_0^i + q^i y^i = e_0^i$, and that $x_0^i > 0$, for all $i = 1, \dots I$.

We define unobservable characteristics to be $(u(\cdot), \xi) = (u(\cdot), (\beta, \pi, R, e))$, (we omit S, since it is fixed), and define demand, as a function of prices, income and these characteristics, as

$$(x_0, y)(q, e_0; u(\cdot), \xi) =$$

$$\begin{array}{ll} \arg\max_{x \geq 0, y} & u(x_0) + \beta \sum_{s=1}^S \pi_s u(x_s), \\ \text{s.t.} & x_0 + qy \leq e_0, \\ & x_s - R_s y \leq e_s, \ \ s = 1, ..., S. \end{array}$$

The following lemma gives necessary and sufficient conditions for the observations to be consistent with expected utility maximization.

Lemma 2. *The following two statements are equivalent:*

1 *There exists fundamentals $(u(\cdot), \xi)$, such that, for all $i = 1, \dots I$,*

$$(x_0^i, y^i) \in (x_0, y)(q^i, e_0^i; u(\cdot), \xi),$$

and such that $e_s + R_s y^i > 0$ for all $s = 1, \dots, S$.

2 *There exists $(\{m^i\}_{i=1}^I, \xi) = (\{m^i\}_{i=1}^I, (\beta, \pi, e, R))$, with $m^i \in \mathbb{R}_{++}^{S+1}$, for $i = 1, \dots, I$, such that*

– *for all* $i = 1, \dots, I$,

$$m_0^i q^i = \beta \sum_s \pi_s R_s m_s^i; \tag{2}$$

– *for all* $i, j = 1, \dots, I$ *and all* $s, s' = 0, \dots S$,

$$(m_s^i - m_{s'}^j)(c_s^i - c_{s'}^j) \leq 0, \quad \textit{and} \quad < 0, \quad \textit{if } c_s^i \neq c_{s'}^j, \tag{3}$$

where $c_s^i = e_s + R_s y > 0$, *for* $s = 1, \dots S$ *and* $c_0^i = e_0^i - q^i y = x_0^i$.

The proof follows directly from Varian (1983). Note that, since asset demand has to satisfy the Strong Axiom of Revealed Preference, these conditions are not vacuous: there exist observations of asset demands and date 0 consumption that cannot be rationalized by any characteristics.

We want to prove that when the number of observations goes to infinity, the set of solutions to the Afriat inequalities from Lemma 2 converges to a singleton which only contains the true underlying characteristic. In order to do so we consider a nested sequence of sets of observations. We denote by $\mathcal{D}^n$ a set of n observed prices and date 0 endowments. We denote by $\mathcal{D}$ an open set of prices and date 0 endowments for which the demand function is well defined and invertible, and as above, denote by $\mathcal{X}_0$ the projection of the open set of observed choices on date 0 consumption.

Theorem 2 (Revealed Preference)**.** *Let* $(\mathcal{D}^n \subset \mathcal{D} : n = 1, \dots)$ *be an increasing sequence of finite sets of observed prices and date 0 endowments with* $\dots, \mathcal{D}^n \subset \mathcal{D}^{n+1}, \dots,$ *and with* $\cup_n \mathcal{D}^n$ *dense in an open set* $\mathcal{D} \subset \mathbb{R}^A \times \mathbb{R}_+$. *Given some underlying characteristics* (u^*, ξ^*) *with* $u^*(1) = 0$ *and* $u^{*'}(1) = 1$ *that satisfy the sufficient conditions of the Identification Theorem, define*

$$(x_0^{i,n}, y^{i,n}) = (x_0, y)(q^{i,n}, e_0^{i,n}; u^*(\cdot), \xi^*), \quad i = 1, \dots, n,$$

and suppose that there is a compact set $\mathcal{K}$, *such that* $(m^{i,n}, \xi^n) \in \mathcal{K}$ *for each* i, n, *and that for each* n, $(\{m^{i,n}\}_{i=1}^n, \xi^n)$ *satisfy (2) and (3) for the observations* $\{(x_0^{i,n}, y^{i,n}), (q^{i,n}, e_0^{i,n})\}_{i=1}^n$. *Then,* $\xi^n \to \xi^*$. *Moreover, if* $u^n(\cdot)$ *is the piece-wise linear function with slopes* $\alpha m_s^{i,n}$ *at* $c_s^{i,n}$ *for all* $(i, n), s$, *with an* $\alpha > 0$ *that ensures the normalization* $u^n(1) = 0$ *and* $u^{n\prime}(1) = 1$, *then* $u^n(x) \to u^*(x)$, *for all* $x \in \mathcal{X}_0$.

Proof. Consider the sequence $((u^n(\cdot), \xi^n) : n = 1, \dots)$, and note that, by compactness, there exists an accumulation point $(\bar{u}(\cdot), \bar{\xi})$. Since $\bar{u}$ must be concave, it must be continuous on $\mathcal{X}_0$. Note that each $(u^n(\cdot), \xi^n)$ as well as $(\bar{u}(\cdot), \bar{\xi})$ correspond to continuous, increasing and concave indirect utility functions, $V^n(x_0, y)$ and $\bar{V}(x_0, y)$, over date 0 consumption and assets.

We first argue that the limit characteristics must generate the same demand function as $(u^*(\cdot), \xi^*)$; that is, for all $(q, e_0) \in \mathcal{D}$,

$$(x_0, y)(q, e_0; \bar{u}(\cdot), \bar{\xi}) = (x_0, y)(q, e_0; u^*(\cdot), \xi^*).$$

If not, there exists $(q^*, e_0^*) \in \mathcal{D}$ and $(x_0^*, y^*) = (x_0, y)(q^*, e_0^*, u^*(\cdot), \xi^*)$ as well as $(\bar{x}_0, \bar{y}) \in \mathbb{R}_+ \times \mathbb{R}^A$ such that $\bar{V}(\bar{x}_0, \bar{y}) > \bar{V}(x_0^*, y^*)$ while $\bar{x}_0 + q^*\bar{y} \leq e_0^*$ and, by the continuity and concavity of $\bar{u}$, without loss of generality,

$$\bar{x}_0 + q^*\bar{y} < e_0^*.$$

Since $\cup_n \mathcal{D}^n \subset \mathcal{D}$ is dense, there exists a sequence $(q^n, e_0^n) \in \mathcal{D}^n : n = 1, \ldots)$, such that $(q^n, e_0^n) \to (q^*, e_0^*)$. By continuity of $u^*(\cdot)$ there is an associated sequence of demands $(x_0^n, y^n) \to (x_0^*, y^*)$.

Since $\bar{V}(\cdot)$ is continuous there is an n sufficiently large such that

$$\bar{V}(x_0^n, y^n) < \bar{V}(\bar{x}_0, \bar{y})$$

and

$$\bar{x}_0 + q^n\bar{y} < e_0^n.$$

But since the sets $\mathcal{D}^n$ are nested, we must have that for all $m \geq n$

$$V^m(y^n) > V^m(\bar{y}),$$

which contradicts the fact that $V^m \to \bar{V}$ point-wise.

To conclude the argument, observe that Lemma 1 implies that, whenever $\bar{u}(\cdot)$ is differentiable, it must coincide with $u(\cdot)$; but since, $\bar{u}(\cdot)$ must be differentiable almost everywhere, it must coincide with $u(\cdot)$ on all of $\mathcal{X}_0$. The Identification Theorem then implies that beliefs must coincide with the true beliefs, and that the accumulation point must be the unique limit of the sequence $((u^n(\cdot), \xi^n) : n = 1, \ldots)$. [2]

□

Note that, in principal, the result gives a method for the construction of beliefs from observed data. However, it is beyond the scope of this paper to investigate the existence of efficient algorithms for the determination of the solution set to the Afriat inequalities.

[2] Mas-Colell (1978) makes the argument in a different setting.

3 Assumptions on fundamentals

In this section, we use the abstract conditions in the Identification Theorem to find conditions on fundamentals that ensure identification. First, examples show that, even when the support of beliefs, (S, R, e), as well as cardinal utility, u, and the discount factor, β, are known, identification may not be possible. Note, that we are concerned with the seemingly much more demanding case, where nothing is known about the beliefs of the individual; nevertheless, it shall turn out that understanding these simple examples provides the key to our general identification results.

1. Suppose there is a single risky asset, second date endowments are 0, $e = 0$, cardinal utility is logarithmic: $u(x) = \ln(x)$, and $\beta = 1$. A simple computation shows that $qy = e_0/2$ – the individual invests a fixed fraction of his wealth in the risky asset, and the demand for the asset is identical for all π; beliefs are not identified.

2. Suppose there is a single, risk-free asset, there is uncertainty about second date endowments, $e \neq 0$, and utility exhibits constant absolute risk aversion, it is CARA, with coefficient of absolute risk aversion equal to 1, i.e. $u(x) = -\exp(-x)$, and $\beta = 1$. Direct computation shows that the demand for the risk-free asset is

$$y = \frac{1}{1+q}\left(e_0 - \ln(q) + \ln(\sum_{s=1}^{S} \pi_s \exp(-e_s)\right);$$

   beliefs are not identified.

There are two obvious ways to solve the problem. One can make assumptions on utility that rule out these cases; or, one can assume that there are several assets available for trade; we shall consider both in detail.

It is useful to note that, with two risky assets and with log-utility identification might still be impossible.

3. Suppose there are two risky assets, there are no endowments, $e = 0$, and $u(x) = \ln(x)$ and $\beta = 1$. Recall that r_{as} is the payoff of asset a in state s. If, for states $s = 1, 2$,

$$\frac{r_{11}}{r_{12}} = \frac{r_{21}}{r_{22}},$$

then $r_{21}/r_{11} = r_{22}/r_{12}$, and the first order conditions that characterise asset demand can be written as

$$\begin{aligned}
\frac{q_1}{\beta}u'(x_0) &= (\pi_1+\pi_2)\frac{1}{\theta_1+\theta_2\frac{r_{21}}{r_{11}}} + \sum_{s=3}^{S}\pi_s r_{1s}u'(x_s),\\
\frac{q_2}{\beta}u'(x_0) &= (\pi_1+\pi_2)\frac{1}{\theta_1\frac{r_{11}}{r_{21}}+\theta_2} + \sum_{s=3}^{S}\pi_s r_{2s}u'(x_s);
\end{aligned}$$

for $s = 1, 2$, beliefs, π_s, cannot be identified separately.

Motivated by this example, and to simplify the exposition, we will from now on focus on the case where a risk-free asset is available for trade. Unfortunately, the following example shows that identification might still be impossible, even if there is a risky and a risk-less asset.

4. Suppose there is a risk-free asset (asset 1) and a risky asset (asset 2). Suppose $e \neq 0$, $u(x) = -\exp(-x)$ and

$$r_{21} = r_{22}, \quad e_1 \neq e_2.$$

The first order conditions that characterize asset demand can be written as

$$\begin{aligned}
q_1 u'(x_0) &= \beta\sum_{s=1}^{S}\exp(-(\theta_1+\theta_2 r_{2s}))\pi_s\exp(-e_s),\\
q_2 u'(x_0) &= \beta\sum_{s=1}^{S}\exp(-(\theta_1+\theta_2 r_{2s}))r_{2s}\pi_s\exp(-e_s);
\end{aligned}$$

for $s = 1, 2$, beliefs, e_s, π_s, cannot be identified separately.

In fact, in this example, identification is impossible even if markets are complete. As we mentioned earlier, existing results on the identification of preferences from demand do not apply when only excess demand is observable, which is the case here: since endowments are unknown, consumption is not observable.

Building on these examples, we now consider two cases. First we assume that there is only a risk-free asset, and we give conditions on cardinal utility that ensure that beliefs can be identified. We then consider the case where there is both a risky and a risk-free asset available for trade and we give conditions on admissible beliefs and cardinal utility which ensure identification.

3.1 Restrictions on cardinal utility

In this section we give conditions on cardinal utility that guarantee the independence of marginal utilities as in the Identification Theorem and allow identification of beliefs only from the observation of demand for the risk-free asset. Note that, in addition to the risk-free asset, there could be risky assets available for trade. Just considering the first order condition for the risk-free asset identifies $e_s + R_s y$ for all s, and varying y then identifies R_s independently of e_s. To simplify the notation we assume in this subsection that there is only a risk-free asset available for trade.

Analogously to the analysis above, given $\mathcal{X}_0$ and $\mathcal{Y}$, define for each ξ^1, ξ^2 $(\mathcal{S}, \bar{e}) = \Sigma(\xi^1, \xi^2)$ and

$$\mathcal{Y}_{\bar{e}} = \{y \in \mathcal{Y} : \bar{e}_s + y \in \mathcal{X}_0 \text{ for all } s \in \mathcal{S}\}.$$

Clearly, the Identification Theorem immediately implies that, if we restrict characteristics to ensure that for all $\xi^1, \xi^2 \in \Xi$, $\mathcal{Y}_{\bar{e}}$ is non-empty, beliefs can be identified if the functions $(u'(\bar{e}_s + y))_{s \in \mathcal{S}}$ are linearly independent over $\mathcal{Y}_{\bar{e}}$. In this section we address the question whether one can find assumptions on cardinal utility that ensure this.

In order to understand why it is difficult to find general necessary and sufficient conditions it is useful to recall that a continuous complex valued function on $\mathbb{R}$ is called mean periodic if it solves the integral equation

$$\int f(x+e)d\mu(e) = 0$$

for some (non-zero) measure with compact support, μ (see Schwartz (1947)). Restricting ourselves to measures with finitely many points in their support it is easy to see that

$$\sum_{i=1}^{K} \alpha_k f(x + e_k) = 0 \tag{4}$$

if $f(x) = \exp(\lambda x)$ and λ is a (complex) root of $\sum_{i=1}^{K} \alpha_k \exp(\lambda e_k) = 0$. Denoting all complex roots by $\lambda_1, ..., \lambda_n$ and denoting by m_j the multiplicity of λ_j it follows that $f(x) = \sum_{j=1}^{n} p_j(x) \exp(\lambda_j x)$ solves (4) whenever each $p_j(.)$ is a polynomial of degree less that m_j. Therefore the real valued solutions to (4) can be both the sum of products of polynomials and the exponential function as well as trigonometric functions and a general characterization is difficult; Remark (iii) in Laczkovich (1986) gives a concrete example.

The following proposition identifies classes of utility functions that allow for identification together with restrictions on beliefs.

Proposition 1. *Beliefs can be identified under any of the following conditions on characteristics.*

(1) For all $\xi^1, \xi^2 \in \Xi$, $\mathcal{Y}_{\bar{e}}$ is non-empty and there exists an analytic function $f : (a,b) \to \mathbb{R}$ with $\mathcal{X}_0 \subset (a,b)$, that coincides with $u'(\cdot)$ on $\mathcal{X}_0$ and that is unbounded in the sense that $\|f(x)\| \to \infty$, as $x \to a$ or as $x \to b$.

(2) Cardinal utility is C^m, $m > 1$, at all but finitely many 'critical points' in $\mathcal{X}_0$, and for each $\xi^1, \xi^2, \in \Xi$, one of the following two conditions holds:

(a) There is a largest 'critical point', $\bar{x} \in \mathcal{X}_0$, at which $u(\cdot)$ is not C^m, and we have

$$\bar{x} - \bar{e}_s \in \mathcal{Y}_{\bar{e}}, \quad \textit{for all } s \in \mathcal{S}.$$

(b) There are (at least) K critical $\bar{x}_i$ at which $u(\cdot)$ is not C^m with

$$\bar{x}_i - \bar{e}_s \in \mathcal{Y}_{\bar{e}}, \quad \textit{for all } s \in \mathcal{S},\ i = 1, 2,$$

all critical points $\bar{x}_j \in \mathcal{X}_0$ satisfy

$$|\bar{x}_i - \bar{x}_j| \neq |\bar{x}_k - \bar{x}_l| \quad \textit{for all } i \neq j, \{i,j\} \neq \{k,l\}$$

and, for all $\xi \in \Xi$, $2S \leq K$.

(3) Cardinal utility is a polynomial of degree n, and for all $\xi \in \Xi$, $S < (1/2)(n+1)$.

Proof. To prove (1), recall that for analytic functions linear independence on any open set and linear independence on the entire domain are equivalent. Suppose $\|f(x)\| \to \infty$ as $x \to a$ (the argument is analogous for $x \to b$). We show that $f(e_1+x), \ldots, f(e_K+x)$ must be linearly independent for all distinct $e_1, ..., e_K$. If they were dependent there would exist $e_1 < e_2 < \ldots < e_K$ and $\alpha \in \mathbb{S}^{K-1}$ such that $\sum_k \alpha_k f(e_k + x) = 0$ for all $x > -e_1 + a$. But, as $x \to -e_1 + a$, $f(e_1 + x) \to \infty$, while all other $f(e_k + x), k > 1$, remain bounded above. There cannot be a linear combination that stays equal to 0 and puts positive weight on $f(e_1 + x)$ – the same argument applies for any e_s.

To prove (2), suppose without loss of generality that $e_1 < ... < e_S$. For (2.a), let $\bar{x}$ denote the largest critical point in $\mathcal{X}_0$ and define $\bar{y}_s \in \mathcal{Y}_{\bar{e}}$ as $\bar{y}_s = \bar{x} - e_s$. If marginal utilities $u'(e_s + y)$ were linearly dependent, without loss of generality we would have

$$u'(e_S + y) = \sum_{s \in \mathcal{S} \setminus \{S\}} \alpha_s u'(e_s + y),$$

which is impossible since the left hand side is not C^m, at $\bar{y}_s$ while the right hand side is. For (2.b), suppose that

$$u'(\bar{e}_1 + y) = \sum_{s \in \mathcal{S} \setminus \{1\}} \alpha_s u'(\bar{e}_s + y).$$

Since there are at least K critical points with $\bar{x}_i - e_1 = \bar{y}_i \in \mathcal{Y}_{\bar{e}}$, $i = 1, ...K$ we must have that there is some state s with where $\bar{y}_l$ and $\bar{y}_k$ are also critical points for some $l, k = 1, ..., K$, implying that there are critical $e_s + \bar{y}_l$ and $e_s + \bar{y}_k$ which have the same distance as $\bar{x}_l$ and $\bar{x}_k$ contradicting the condition in the proposition.

To prove (3), recall that the Wronskian matrix of a function $f : \mathbb{R} \to \mathbb{R}^n$ is defined as

$$W = \begin{pmatrix} f_1 & \dots & f_i & \dots & f_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_1^{(k)} & \dots & f_i^{(k)} & \dots & f_n^{(k)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)} & \dots & f_i^{(n-1)} & \dots & f_n^{(n-1)} \end{pmatrix};$$

If the utility function is polynomial,

$$u(x) = a_0 + a_1 x + \dots, a_l x^l + \dots a^n x^n,$$

then

$$u^{(l)}(x) = a_l l! + \dots a_k \frac{k!}{(k-l)!} x^{k-l} + \dots a_n \frac{n!}{(n-l)!} x^{n-l}, \quad l = 0, \dots n,$$

and, in particular,

$$u^n(x) = a_n n!.$$

To prove that W is non-singular, consider the matrix

$$A_n = \begin{pmatrix} a_1 & \dots & (1+k)a_{k+1} & \dots & \dots & \dots & \dots & na_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_l l! & \dots & a_{l+k}\frac{(l+k)!}{k!} & \dots & a_n \frac{n!}{(n-l)!}x^{n-l} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_n n! & \dots & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix},$$

the submatrix

$$A_{S,n} = \begin{pmatrix} a_{n-S+1}(n-S+1)! & \dots & a_n \frac{n!}{(S-1)!}x^{(S-1)} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_n n! & \dots & 0 & 0 & \dots \end{pmatrix},$$

and the matrix

$$B_n^S = \begin{pmatrix} \dots & 1 & \dots \\ \vdots & \vdots & \vdots \\ \dots & (e_s+x)^k & \dots \\ \vdots & \vdots & \vdots \\ \dots & (e_s+x)^{(S-1)} & \dots \\ \vdots & \vdots & \vdots \\ \dots & (e_s+x)^{n-1} & \dots \end{pmatrix}.$$

The Wronskian of the family of functions $\{u^{(n-S+1)}(e_s+x)\}$, that is, of the derivatives of order $(n-S+1)$ of the functions $\{u(e_s+x)\}$, is

$$W_{(n-S+1)} = A_{S,n} B_n^S =$$

$$\begin{pmatrix} a_{n-S+1}(n-S+1)! & \dots & a_n \frac{n!}{(S-1)!} x^{(S-1)} \\ \vdots & \vdots & \vdots \\ a_n n! & \dots & 0 \end{pmatrix} \begin{pmatrix} \dots & 1 & \dots \\ \vdots & \vdots & \vdots \\ \dots & (e_s+x)^k & \dots \\ \vdots & \vdots & \vdots \\ \dots & (e_s+x)^{(S-1)} & \dots \end{pmatrix},$$

a square matrix of dimension $S \times S$, that is invertible: it is the product of two square matrices, of which the first term is upper-diagonal, with non-vanishing terms on the diagonal, $a_n n! = u_{(n)}(x) \neq 0$, while the second is the Vandermonde matrix of the random variables $\{(e_s+x)\}$. Since the Wronskian has full rank beliefs can be identified.

□

The result provides a large class of cardinal utility functions that ensure that beliefs can be identified. The class of analytic functions that satisfy an Inada condition is perhaps most relevant for many applications since it subsumes all utility functions that exhibit constant relative risk aversion. If cardinal utility is piecewise polynomial the proposition shows that beliefs can be identified if demand is observed globally or if there are sufficiently many distinct pieces in the observed range of date 0 consumption. If cardinal utility is polynomial, beliefs can be identified if the degree of the polynomial is sufficiently high relative to the maximal number of possible states.

The possibility of identification for polynomial cardinal utility raises the question of what happens if we pass to the limit. That is, why can beliefs not be identified for the function $u'(x) = \exp(-x)$ although this function can be expressed as a power series. We give a (partial) answer to this question in the appendix.

Generic identification Since identification fails only for mean-periodic functions, it should be possible to guarantee identification generically, at least for a finite state space and given endowments. Let $\mathcal{A}$ be a finite dimensional family of cardinal utility functions sufficiently rich in perturbations: if $u(\cdot) \in$

$\mathcal{A}$, then $\tilde{u}(x) = u(x) + \sum_{k=0}^{n} a_k x^k \in \mathcal{A}$, for $a_k \in (-\varepsilon, \varepsilon)$, for some $\varepsilon > 0$, and any finite n.

For simplicity of exposition, let $S = 2$ and consider the function $F : \mathbb{S}^{S-1} \times (-\varepsilon, \varepsilon)^n \to \mathbb{R}^2$, for some $n \geq 3$, defined by

$$F(\theta_1, \theta_2, \ldots, a_k, \ldots) = (\theta_1, \theta_2)W = (\theta_1, \theta_2)AB,$$

where

$$A = \begin{pmatrix} a_1 & 2a_2 & 3a_3 & 4a_4 & \ldots \\ \\ 2a_2 & 6a_3 & 12a_4 & \ldots & \ldots \end{pmatrix},$$

and

$$B = \begin{pmatrix} 1 & 1 \\ \\ (e_1 + x) & (e_2 + x) \\ \\ (e_1 + x)^2 & (e_2 + x)^2 \\ \\ \vdots & \vdots \\ \\ \end{pmatrix}.$$

By a direct computation,

$$D_{a_1,a_2,a_3}F = \begin{pmatrix} \theta_1 & 2\theta_2 + 2\theta_1(e_1 + x) & 6\theta_2(e_1 + x) + 3\theta_1(e_1 + x)^2 \\ \\ \theta_1 & 2\theta_2 + 2\theta_1(e_2 + x) & 6\theta_2(e_2 + x) + 3\theta_1(e_2 + x)^2 \end{pmatrix}.$$

Since $(\theta_1, \theta_2) \in \mathbb{S}^1$, while $e_1 \neq e_2$, the matrix $D_{a_1,a_2,a_3}F$ has full row rank, which extends to the matrix DF. It follows that $F \pitchfork 0$, and, by the transversal density Theorem, $F_u : \mathbb{S}^1 \pitchfork 0$, for u in an subset of $\mathcal{A}$ of full Lebesgue measure. Then, since $\dim \mathbb{S}^1 < 2$, there is no (θ_1, θ_2) such that $F(\theta_1, \theta_2, \ldots, a_k, \ldots) = (\theta_1, \theta_2)AB = 0$. It follows that AB, the Wronskian, is of full rank, which allows for identification.

The argument takes the endowment of the individual as given and known. It extends to the case in which endowments lie in a finite set.

3.2 A risky asset separates all uncertainty

In this subsection, we assume that there are two assets available for trade, a risk-free asset (a= 1) and a risky asset (a = 2). It is without loss of generality to focus, as above, on the case of a single risky asset.

It is useful to first consider the situation where the risky asset defines all uncertainty: that is, all admissible beliefs can be described as beliefs over asset payoffs, and there is a function from asset payoffs to individual endowments. The case of individual endowments being 0 at all states $s = 1, \ldots, S$ is obviously a special case. Another special case is the case of neoclassical production, where shocks to total factor productivity determine both wages (endowments) and returns to capital (payoff of a risky asset).

When the risky asset defines all uncertainty the beliefs over the asset payoffs can be identified if cardinal utility is smooth with non-vanishing derivatives. We then relax this assumption, and merely require that the payoff of the risky asset separates uncertainty: for all possible beliefs, $R_s \neq R_{s'}$, if $s \neq s'$. This assumption is strictly weaker since it could be the case that different fundamentals have the same asset payoffs, but different endowments. Example 3 above implies that if the individual has CARA utility identification is no longer possible. It turns out that the condition needed for identification is somewhat intricate. Only ruling out CARA utility does not suffice to ensure that the result is restored.

Proposition 2. *Suppose there exist a risk-free asset, $r_1(s) = 1$, for all s, and asset 2 separates all uncertainty; that is, for all possible beliefs,*

$$r_2(s) \neq r_2(s'), \quad \textit{for all} \;\; s \neq s'.$$

Suppose that for all characteristics $\xi^1, \xi^2 \in \Xi$, $S^1, S^2 \leq \bar{S}$, for some $\bar{S} \leq \infty$, and that the joint support $(\mathcal{S}, \bar{R}, \bar{e}) = \Sigma(\xi^1, \xi^2)$ is such that there exists a $\bar{y} \in \mathcal{Y}_{\bar{e},\bar{R}}$, with $u(e_s + R_s y)$ having $2\bar{S}$ non-vanishing derivatives on a neighborhood around $\bar{y}$. Then, the demand function for consumption and assets identifies the unobservable characteristics (u, ξ) if one of the two following assumptions holds:

(1) There is some function $f : \mathbb{R} \to \mathbb{R}_+$, such that, for any possible individual characteristics $(S, \beta, \pi, R, e) \in \Xi$,

$$e(s) = f(r_2(s)) \quad s = 1, ..., S.$$

(2) There is no $1 \leq k \leq 2\bar{S}$ such that the k'th derivative of cardinal utility can be written as

$$u^{(k)}(x) = p(x) \exp(\alpha x), \tag{5}$$

for some periodic function $p(\cdot)$ and some $\alpha \in \mathbb{R}$.

Proof. As in the proof of the Identification Theorem, suppose that characteristics ξ^1 and ξ^2 rationalize observed asset demand and let $(\mathcal{S}, \bar{R}, \bar{e}) = \Sigma(\xi^1, \xi^2)$. Without loss of generality let $\mathcal{S} = \{1, \ldots, K-1, K\}$ for some $K > S^1$. It suffices to prove that the functions $\{u'(\bar{e}_s + y_1 + y_2 \bar{r}_{2s})\}_{s \in \mathcal{S}}$, are differentiably linearly independent. For this take the differential operators that define $\tilde{W}$ in (1) to be

$$\Delta_{i_j} = (\frac{\partial}{\partial y_1})^{K-j}(\frac{\partial}{\partial y_2})^{j-1}.$$

We obtain

$$\tilde{W} = \begin{pmatrix} u^{(K)}(\bar{e}_1 + y_1 + y_2\bar{r}_{21}) & \dots & u^{(K)}(\bar{e}_K + y_1 + y_2\bar{r}_{2K}) \\ \vdots & \vdots & \vdots \\ u^{(K)}(\bar{e}_1 + y_1 + y_2\bar{r}_{21})\bar{r}_{21}^{K-1} & \dots & u^{(K)}(\bar{e}_K + y_1 + y_2\bar{r}_{2k})\bar{r}_{2K}^{K-1} \end{pmatrix}.$$

Since the Vandermonde matrix

$$\begin{pmatrix} 1 & \dots & 1 \\ \bar{r}_{21} & \dots & \bar{r}_{2K} \\ \vdots & \vdots & \vdots \\ \bar{r}_{21}^{K-1} & \dots & \bar{r}_{2K}^{K-1} \end{pmatrix}$$

has full rank, the matrix $\tilde{W}$ has full rank and the Identification Theorem implies that beliefs can be identified under (1).

Under (2), suppose linear independence fails and there is an $\alpha \in \mathbb{S}^{K-1}$ such that $\sum_{s=1}^{K} \alpha_s u'(\bar{e}_s + y_1 + y_2\bar{r}_{2s}) = 0$ on a neighborhood around $\bar{y}$. Since in both characteristics ξ^1 and ξ^2 the payoff of asset 2 separates uncertainty, each $\bar{r}_{2s}$ can be identical across at most two states. Theorem 1 in Laczkovich (1986) implies that if for some $\alpha, e \in \mathbb{R}$ a real valued (measurable) function $f : \mathcal{A} \subset \mathbb{R} \to \mathbb{R}$ satisfies $f(x) = \alpha f(x + e)$ for all $x \in \mathcal{A}$ it must be of the form (5). Therefore condition (2) implies that for all s and s' with $\bar{r}_{2s} = \bar{r}_{2s'}$ we have that for all $1 \leq k \leq K$, $\alpha_s u^{(k)}(\bar{e}_s + y_1 + y_2\bar{r}_{2s}) + \alpha_{s'} u^{(k)}(\bar{e}_{s'} + y_1 + y_2\bar{r}_{2s'}) \neq 0$ whenever $(\alpha_s, \alpha_{s'}) \neq 0$. Taking L to be the number of distinct asset payoffs in $\{\bar{r}_{21}, \dots, \bar{r}_{2K}\}$, and defining the $L \times L$ matrix $\tilde{W}$ by taking the entry $(\alpha_s u^{(K)}(\bar{e}_s + y_1 + y_2\bar{r}_{2s}) + \alpha_{s'} u^{(K)}(\bar{e}_{s'} + y_1 + y_2\bar{r}_{2s'}))\bar{r}_{2s}^{j}$ instead of $u^{(K)}(\bar{e}_{s'} + y_1 + y_2\bar{r}_{2s'}))\bar{r}_{2s'}^{j}$ and $u^{(K)}(\bar{e}_s + y_1 + y_2\bar{r}_{2s}))\bar{r}_{2s}^{j}$ whenever $r_{2s} = r_{2s'}$ we again obtain a matrix of full rank and therefore there cannot be $\alpha \in \mathbb{S}^{K-1}$ such that $\sum_{s=1}^{K} \alpha_s u'(\bar{e}_s + y_1 + y_2\bar{r}_{2s}) = 0$ on a neighborhood around $\bar{y}$. The functions $\{u'(\bar{e}_s + y_1 + y_2\bar{r}_{2s})\}$ are linearly independent and beliefs can be identified.

□

One might wonder why we assume that all derivatives of $u'(\cdot)$ are not of the form (5) and why it is not enough to assume this only for $u'(\cdot)$ itself.

A simple example illustrates the problem. Suppose $u'(x) = \exp(-x) + 1$. Clearly this function is not of the form (5) and in fact one can easily verify that $u'(x)$ and $u'(a+x)$ are linearly independent for all $a \neq 0$. However, beliefs cannot be identified with this utility function. The necessary and sufficient first order conditions that define the asset demand function can be written as

$$\begin{aligned} q_1 u'(x_0) &= \beta \left(1 + \sum_{s=1}^{S} \exp(-(\theta_1 + \theta_2 r_{2s}))\pi_s \exp(-e_s) \right), \\ q_2 u'(x_0) &= \beta \left(\sum_{s=1}^{S} \pi_s r_{2s} + \sum_{s=1}^{S} \exp(-(\theta_1 + \theta_2 r_{2s})) r_{2s} \pi_s \exp(-e_s) \right). \end{aligned}$$

Suppose for two characteristics ξ^1, ξ^2 we have $S^1 = S^2 = 3$, $R^1 = R^2$ and $e^1 \neq e^2$. As long as $\beta^1 = \beta^2$ and π^1, π^2 satisfy

$$\sum_{s=1}^{3} \pi_s^1 r_{2s} = \sum_{s=1}^{3} \pi_s^2 r_{2s} \quad \text{and} \quad \pi_s^1 e_s^1 = \pi_s^2 e_2^2 \text{ for all } s = 1, \ldots, S;$$

the two characteristics generate identical asset demand.

4 Extensions

We show that part of the analysis remains valid if utility is not stationary, and cardinal utility at date 1 may differ from utility at date 0. And we consider a multi-period model, where cardinal utility is stationary but only demand at date 0 is observable.

4.1 Non-stationary utility

We now assume that the utility function need not be stationary and cardinal utility at date 1 could be distinct from the utility at date 0; that is,

$$U(x_0, \ldots, x_s, \ldots) = u(x_0) + \sum_{s=1}^{S} \pi_s v(x_s),$$

While Lemma 1 still holds and one can recover $u(\cdot)$, this does not provide us with any information about the function $v(\cdot)$. Nevertheless, it is straightforward to extend the Identification Theorem to cover this case. A characteristics, now, include the date 1 cardinal utility $v(\cdot)$. Given two such (extended) characteristics $\xi^1 = (v^1(\cdot), S^1, \pi^1, R^1, e^1)$ and $\xi^2 = (v^2(.), S^2, \pi^2, R^2, e^2)$ with

$v^1(\cdot) \neq v^2(\cdot)$ on some subset of the range of possible consumption identification is possible if the functions $\{v^{1\prime}(e_s^1 + R_s^1 y)\}_{s=1}^{S^1}$ together with $\{v^{2\prime}(e_s^2 + R_s^2 y)\}_{s=1}^{S^2}$ are linearly independent. If this is the case the argument from the proof of the Identification Theorem can be applied analogously and beliefs can be identified.

However, with unknown labor endowments the functions will generally be linearly dependent. Suppose, as an example, that $S^1 = 1$ and $S^2 = 2$, and there is only one, risk-free asset. It is then clear that if $v^1(x + e_1^1) = \pi_1^2 v^2(x + e_1^2 - e_1^1) + \pi_2^2 v^2(x + e_2^2 - e_1^1)$ the two characteristics will generate identical demand. As in Example 3 above, it is clear that the presence of a risky asset will not help as long as the risky asset does not define uncertainty as in Proposition 2(1).

Moreover, if $v(\cdot)$ is not analytic, it is difficult to rule out the case of linear dependence simply because given $v^1(\cdot)$ one can always define another cardinal utility, $v^2(\cdot)$, by

$$v^{1\prime}(e_s + yR_s) = \gamma_s v^{2\prime}(e_s + yR_s), \quad s = 1, \ldots S,$$

for some $(\gamma_s)_{s=1}^S$ that ensure that $\sum_s \gamma_s \pi_s^2 = 1$. If one only observes asset demand locally, as in the Identification Theorem, it is not possible to identify beliefs separately from $\gamma_1, \ldots, \gamma_S$.

When utility is assumed to be analytic and the risky asset defines the uncertainty beliefs can be identified.

Proposition 3. *Suppose $v(\cdot)$ is analytic and has non-vanishing derivatives of all orders. Suppose there exist a risk-free asset, $r_{1s} = 1$, for all s, and asset 2 separates all uncertainty; that is, for all possible beliefs,*

$$r_{2s} \neq r_{2s'} \quad \textit{for all} \quad s \neq s',$$

and there is some function $f : \mathbb{R} \to \mathbb{R}_+$, such that, for any possible individual characteristics $\xi = (v(\cdot), S, \pi, e, R) \in \Xi$,

$$e_s = f(r_{2s}) \quad s = 1, ..., S.$$

Then the demand function for consumption and assets identifies the unobservable characteristics.

Proof. Analogously to the proof of the Identification Theorem, suppose that characteristics $\xi^1 = (v^1(\cdot), S^1, \pi^1, R^1, e^1)$ and $\xi^2 = (v^2(\cdot), S^2, \pi^2, R^2, e^2)$ rationalize observed asset demand. As above, define $\bar{e}_s = e_s^1, \bar{R}_s = R_s^1$, $\bar{v}_s(\cdot) = v^1(\cdot)$ for $s = 1, \ldots, S^1$ and $\bar{e}_{s+S^1} = e_s^2, \bar{R}_{s+S^1} = R_s^2$, $\bar{v}_{s+S^1}(\cdot) = v^2(\cdot)$ for $s = 1, \ldots, S^2$. Let $K = S^1 + S^2$. First assume that the K'th derivatives of

v^1 and v^2 are independent, i.e. there is no γ such that $v^{1(K)}(c) = \gamma v^{2(K)}(c)$ for all c in an open neighborhood. To show identification, it suffices to show that there is no $\alpha \in \mathbb{S}^{K-1}$ with $\sum_{s=1}^{K} \alpha_s \bar{v}'_s(\bar{e}_s + y\bar{R}_s) = 0$. As in the proof of Proposition 2, define the differential operators in the matrix $\tilde{W}$ in (1) to be $\Delta_{i_j} = (\frac{\partial}{\partial y_1})^{K-j}(\frac{\partial}{\partial y_2})^{j-1}$. As in the proof of Proposition 2(2), combine the derivatives of $\bar{v}_s(\bar{e}_s + y\bar{R}_s)$ and $\bar{v}_{s'}(\bar{e}_{s'} + y\bar{R}_{s'})$ into a single element of the matrix whenever $\bar{R}_s = \bar{R}_{s'}$. The non-singularity of the Vandemonde matrix implies that whenever $\alpha_s \bar{v}_s^{(K)}(e_s + y\bar{R}_s) \neq \alpha_{s'} \bar{v}_{s'}^{(K)}(e_{s'} + y\bar{R}_{s'})$ we must have $\sum_{s=1}^{K} \alpha_s \bar{v}'_s(\bar{e}_s + y\bar{R}_s) \neq 0$ and hence identification is possible.

If there is a γ such that $v^{1(K)}(\cdot) = \gamma v^{2(K)}(\cdot)$, the derivative of order $(K-1)$ of the first order condition with respect to the demand for the risk-free asset for a fictitious investor who faces states $s = 1, \ldots, K$ and has state dependent utility $\hat{v}_s(\cdot)$ defined as

$$\hat{v}_s(x) = \begin{cases} v^1(x) & \text{if } s \leq S^1 \text{ and } R_s \notin \{R_1^2, \ldots, R_{S^2}^2\}, \\ \\ v^1(x) + v^2(x) & \text{if } s \leq S^1 \text{ and } R_s \in \{R_1^2, \ldots, R_{S^2}^2\}, \\ \\ v^2(x) & \text{otherwise,} \end{cases}$$

is given by

$$\beta \sum_{s \in \mathcal{S}} \pi_s \hat{v}_s^{(K)}(\bar{e}_s + y\bar{R}_s) = \frac{\partial^{(K-1)}[qu'(x_0)]}{\partial y_1^{(K-1)}},$$

where $\mathcal{S}$ is defined as in Proposition 2. By the same argument as in the proof of Proposition 2, beliefs for this fictitious investor are uniquely identified. But since $v^{1(i)}(x) = \gamma v^{2(i)}(x)$ for all $i \geq K$ only one of the characteristics ξ^1, ξ^2 is consistent with these beliefs. Hence only one of these characteristics can generate the observed demand function.

□

4.2 Multiple periods

Another obvious question is how our results extend to a multi-period setting. Suppose the individual maximizes a time-separable subjective expected utility over $T+1$ periods,

$$U(x) = u(x(0)) + \sum_{t=1}^{T} \beta^t \sum_{s^t \in \Sigma} \pi(s^t) u(x(s^t)), \quad x \geq 0,$$

where $s^t = (s_1, \ldots, s_t)$ is a sequence of realizations of shocks, s_t up to date t. Assets, $a = 1, \ldots, A$, of one period maturity are traded in dates $t = 0, \ldots (T-$

1). Pay-offs of an asset traded at s^{t-1} are $r_a(s^t)$ and payoffs across assets are $R(s^t) = (r_1(s^t), \ldots, r_a(s^t), \ldots, r_A(s^t))$. Holdings of assets are $y(s^t) = (\ldots, y_a(s^t), \ldots)^\top$.

To use our insights from the two date problem, it is useful to write the agent's problem recursively. We write $s^{t+1} \succeq s^t$ for an immediate successor of s^t at $t+1$, and we define

$$v_{s^T}(y) = u(e(s^T) + R(S^t)y) \text{ and } d_{s^T} = e(s^T),$$

and recursively for up to $t = 1$,

$$\begin{array}{rcl} v_{s^t}(y) & = & \max_{y'} u(e(s^t) + R(s^t)y - q(s^t)y') + \beta \sum_{s^{t+1} \succeq s^t} \frac{\pi(s^{t+1})}{\pi(s^t)} v_{s^{t+1}}(y'), \\ \text{s.t.} & & R(s^{t+1})y' + d_{s^{t+1}} \geq 0 \quad s^{t+1} \succeq s^t, \end{array}$$

and

$$d_{s^t} = \max_{y'} e(s^t) + q(s^t)y' \text{ s.t. } R(s^{t+1})y' + d_{s^{t+1}} \geq 0 \quad s^{t+1} \succeq s^t.$$

While in a two date setting it is natural to assume that only asset demand in the first date is observable, in this multiple date setting one can consider various different cases. We assume that one only observes how the demand for assets at date 0 changes as prices and incomes change at date 0. Of course, in an equilibrium setting, price changes in one period most likely lead to an agent revising his expectations on prices in subsequent periods. However, as we pointed out in the introduction, we focus on a 'partial equilibrium' situation where prices and incomes change today and this has no effect on an agent's belief over future outcomes. As in the rest of the paper we assume that the demand function at date 0 is observed in an open neighborhood of prices and endowments. Consistent with our earlier notation we define $v_s(y)$, for $s = 1, ..., S$, to be the possible value functions at $t = 1$.

For simplicity we assume that cardinal utility, $u(\cdot)$, is analytic. Lemma 1 then directly implies that $u(\cdot)$ can be identified on $(0, \infty)$. It is useful to write the unknown fundamentals as $\xi = (\beta, \xi_1, ..., \xi_S) \in \Xi$ where ξ_s specifies all subsequent endowments, payoffs of assets and probabilities. With this we can write $v_s(y) = v(y|\xi_s)$ for some state invariant function $v(.)$ which is known once $u(\cdot)$ is known. We denote by $v'(y) = \partial v(y|\xi)/\partial y_1$, where it is assumed, as before, that asset 1 is a risk-free bond, paying 1 in all states.

We now redefine $(\mathcal{S}, \bar{\xi}) = \Sigma(\xi^1, \xi^2)$ as $\bar{\xi}_s = \xi^1_s$ for $s \leq S$ and $\bar{\xi}_s = \xi^2_{s-S}$, $\bar{R}_s = R^2_{s-S}$ for $s > S$, as well as

$$\mathcal{S} = \{1, \ldots, S\} \cup \left\{ s \in \{S, \ldots, 2S\} : \bar{\xi}_s \notin \{\xi_1, \ldots, \xi_S\} \right\}.$$

The Identification Theorem, then, translates to the following result:

Proposition 4. *The demand function identifies the unobservable characteristics $\xi \in \Xi$ if, for any $\xi^1, \xi^2 \in \Xi$, the joint support $(\mathcal{S}, \bar{\xi}) = \Sigma(\xi^1, \xi^2)$ is such that the functions $\{v'(y|\bar{\xi}_s)\}$ are linearly independent on $\mathcal{Y}_{(\bar{e},\bar{R})}$.*

Conversely, if there are characteristics $\xi \in \Xi$ for which $\{v(y|\xi_s)\}$ are not linearly independent on $\mathcal{Y}$, then identification is impossible.

Under stationarity assumptions it might be possible to derive conditions that ensure the required independence. This is subject to further research.

5 Concluding remarks

Here, identification proceeds from an observable demand function. Though common practice, this is problematic. It is an obvious question whether our approach has any implications on equilibrium prices. While it is beyond the scope of this paper to provide a detailed answer to this, it is useful to point out that the arguments in Chiappori, Ekeland, Kubler, and Polemarchakis (2004) can be applied here and it can be shown that, under appropriate conditions on cardinal utilities, the map from profiles of individual endowments in assets and period 0 commodity to equilibrium asset prices identifies beliefs. In this formulation assets are productive (trees), and the, $r_{a,s}$ are output, wealth or consumption; and, individuals are endowed with assets, f_0^i. Equilibrium prices, then, satisfy $\sum_i (x_0^1(q, e_0^i, f_0^i), y^i(q, e_0^i, f_0^i)) = \sum_i (e_0^i, f_0^i)$, and the set equilibria of equilibria as endowments vary is $\mathcal{W} = \{(q, \overrightarrow{(e_0^i, f_0^i)}) : \sum_i (x_0^i(q, e_0^i, f_0^i), y^i(q, e_0^i, f_0^i)) = \sum_i (e_0^i, f_0^i)\}$. Under smoothness assumptions, the equilibrium set has a differentiable manifold structure, and it identifies aggregate demand locally. The argument in Chiappori, Ekeland, Kubler, and Polemarchakis (2004), in an abstract context that applies here, is that the aggregate demand identifies individual demand as long as the latter satisfies a rank condition on wealth effects following Lewbel (1991).

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Appendix: Wronskians and power series

For power series, it is instructive to consider the case of CARA cardinal utility,

$$u(x) = -e^{-x} = -1 + x + \ldots + (-1)^{(k+1)}\frac{1}{k!}x^k + \ldots (-1)^{(n+1)}\frac{1}{n!}x^n, \ldots;$$

In order to simplify the exposition, we restrict attention to the case $S = 2$. The polynomial approximation of $u(x)$ of order n is

$$u_n(x) = -1 + x + \ldots + (-1)^{(k+1)}\frac{1}{k!}x^k + \ldots (-1)^{(n+1)}\frac{1}{n!}x^n;$$

evidently,

$$u_n^{(1)}(x) = 1 - x + \ldots + (-1)^{(k+1)}\frac{1}{(k-1)!}x^{(k-1)} + \ldots (-1)^{(n+1)}\frac{1}{(n-1)!}x^{(n-1)},$$

and

$$u_n^{(2)}(x) = -1 + x + \ldots + (-1)^{(k+1)}\frac{1}{(k-2)!}x^{(k-2)} + \ldots (-1)^{(n+1)}\frac{1}{(n-2)!}x^{(n-2)}.$$

It follows that, if

$$A^{2,n} = \begin{pmatrix} 1 & -1 & \ldots & (-1)^{k+1}\frac{1}{(k-1)!} & \ldots & \ldots & (-1)^{(n+1)}\frac{1}{(n-1)!} \\ & & & & & & \\ -1 & 1 & \ldots & (-1)^{k+2}\frac{1}{(k-1)!} & \ldots & (-1)^{n+1}\frac{1}{(n-2)!} & 0 \end{pmatrix},$$

and

$$B_n^2 = \begin{pmatrix} 1 & 1 \\ & \\ (e_1 + x) & (e_2 + x) \\ & \\ \vdots & \vdots \\ & \\ (e_1 + x)^k & (e_2 + x)^k \\ & \\ \vdots & \vdots \\ & \\ (e_1 + x)^{(n-1)} & (e_2 + x)^{(n-1)} \end{pmatrix},$$

the Wronskian of the family of functions $\{u^{(n-S+1)}(e_s + x)\}$, that is, of the derivatives of order $(n-S+1)$ of the functions $\{u(e_s+x)\}$, is

$$W_{2,n} = A_n^2 B_n^2 =$$

$$\begin{pmatrix} 1 & -1 & \dots & \dots & (-1)^{(n+1)}\frac{1}{(n-1)!} \\ -1 & 1 & \dots & (-1)^{n+1}\frac{1}{(n-2)!} & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ (e_1+x) & (e_2+x) \\ \vdots & \vdots \\ (e_1+x)^k & (e_2+x)^k \\ \vdots & \vdots \\ (e_1+x)^{(n-1)} & (e_2+x)^{(n-1)} \end{pmatrix}.$$

For finite n, Proposition 1(3) implies that the rank of $W_{2,n} = A_n^2 B_n^2$ is full, even if this is not clear from the expressions above. But, as $n \to \infty$, the matrix A_n^2 converges to a matrix of row rank 1, which implies that the Wronskian is singular; this accounts for the failure of identification of CARA cardinal utility.

Alternatively, for CRRA cardinal utility, and, in particular,

$$u(x) = \ln x,$$

the power series expansion at $\bar{x} = 1$ is

$$u(x) = \ln x = 0+(x-1)+\ldots+(-1)^{(k-1)}\frac{1}{k}(x-1)^k+\ldots(-1)^{(n-1)}\frac{1}{n}(x-1)^n,\ldots;$$

In order to simplify the exposition, we restrict attention to the case $S = 2$. The polynomial approximation of $u(x)$ of order n is

$$u_n(x) = 0+(x-1)+\ldots+(-1)^{(k-1)}\frac{1}{k}(x-1)^k+\ldots(-1)^{(n-1)}\frac{1}{n}(x-1)^n;$$

evidently,

$$u_n^{(1)}(x) = 1-x+\ldots+(-1)^k(x-1)^k+\ldots+(-1)^{(n-1)}x^{(n-1)},$$

and

$$u_n^{(2)}(x) = -1+x+\ldots+(-1)^{(k+1)}(k+1)(x-1)^k+\ldots+(-1)^{(n-1)}(n-1)x^{(n-2)}.$$

It follows that

$$A_n^2 = \begin{pmatrix} 1 & -1 & \dots & (-1)^k & \dots & \dots & (-1)^{(n-1)} \\ -1 & 2 & \dots & (-1)^{k+1}(k+1) & \dots & (-1)^{n-1} & 0 \end{pmatrix},$$

$$B_n^2 = \begin{pmatrix} 1 & 1 \\ (e_1+x-1) & (e_2+x-1) \\ \vdots & \vdots \\ (e_1+x-1)^k & (e_2+x-1)^k \\ \vdots & \vdots \\ (e_1+x-1)^{(n-1)} & (e_2+x-1)^{(n-1)} \end{pmatrix},$$

and the Wronskian of the family of functions $\{u^{(n-S+1)}(e_s+x)\}$ is

$$W_{2,n} = A_n^2 B_n^2 =$$

$$\begin{pmatrix} 1 & -1 & \dots & \dots & (-1)^{(n+1)}\frac{1}{(n-1)!} \\ -1 & 2 & \dots & (-1)^{n+1}\frac{1}{(n-2)!} & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ (e_1+x-1) & (e_2+x-1) \\ \vdots & \vdots \\ (e_1+x-1)^k & (e_2+x-1)^k \\ \vdots & \vdots \\ (e_1+x-1)^{(n-1)} & (e_2+x-1)^{(n-1)} \end{pmatrix}.$$

For all n, and as $n \to \infty$, the matrix A_n^2 remains of rank 2; this is in contrast to the CARA case.