

National Centre of Competence in Research  
Financial Valuation and Risk Management

Working Paper No.604

## Three Solutions to the Pricing Kernel Puzzle

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First version: March 2010  
Current version: June 2011

This research has been carried out within the NCCR FINRISK project on  
“Behavioural and Evolutionary Finance”

# Three Solutions to the Pricing Kernel Puzzle\*

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June 17, 2011

## Abstract

The pricing kernel is an important link between economics and finance. In standard models of financial economics it is proportional to the aggregate utility in the economy. These models have complete markets and risk-averse agents with correct beliefs. Consequently, the pricing kernel in these models is a decreasing function of aggregate resources. However, there is ample empirical evidence that the pricing kernel has some increasing parts, which is the so called pricing kernel puzzle. In this paper we first show that neither of the three assumptions is needed for the pricing kernel to be generally decreasing and we show then that if at least one of the three assumptions is violated, the pricing kernel can have increasing parts. We explain the economic principles that lead to the increasing part in the pricing kernel. In order to check the empirical relevance of the different possible explanations, the resulting pricing kernels are then compared with the empirical pricing kernel estimated in Jackwerth (2000).

**Keywords:** Pricing kernel puzzle; Financial market equilibrium; Risk-seeking behaviour; Biased beliefs; Incomplete markets

**JEL classification:** D53,G12

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\*We thank two anonymous referees, Bernard Dumas, Piero Gottardi, Jens Jackwerth, Michael Magill, Maria Putintseva, Martine Quinzii, Martin Schweizer and the participants of the EWGET, the NCCR FINRISK conference in Gerzensee and the Seminars in Berlin, Davis, Karlsruhe, Konstanz, San Diego, San Jose and Zurich for helpful comments. Jens Jackwerth graciously provided the density estimates used for the calibrations. Financial support by the National Centre of Competence in Research “Financial Valuation and Risk Management” (NCCR FINRISK) is gratefully acknowledged.

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# 1 Introduction

The pricing kernel is the ratio of the state prices implied by asset returns and the physical probabilities driving those returns. It is an important link between economics and finance since it provides the connection between asset prices and fundamental economic principles like scarcity of resources and decreasing marginal utility of wealth. Indeed in standard models of financial economics the pricing kernel is proportional to the marginal utility of a representative investor. In such models risk averse investors with correct beliefs trade in a complete set of markets. Since in these models the first welfare theorem holds, asset prices can be derived by a single decision problem of a so called representative investor. Moreover, the representative investor's utility is a weighted sum of the individual utilities and thus also has correct beliefs and displays risk aversion (Negishi, 1960; Constantinides, 1982). Consequently, the pricing kernel should be positive and decreasing as a function of aggregate resources. Empirical studies show that the pricing kernel is positive and generally decreasing but also has increasing parts. The latter is known as the pricing kernel puzzle. Is the pricing kernel puzzle an example for Shiller's (2000; 2005) claim that asset prices have a life on their own that cannot be explained by economics? This is an important question at the center of economics and finance.

First we show that even in incomplete markets and allowing for risk-seeking behaviour and heterogeneous beliefs, the pricing kernel is positive and generally decreasing which means that affine functions fitting the pricing kernel must be decreasing. Thus the scarcity of resources still shapes the pricing kernel and over all the decreasing marginal utility of wealth has some bite. This leaves us with the puzzling observation that for some empirically very relevant parts the pricing kernel is observed to be increasing.

The pricing kernel puzzle points out that the standard model is too restrictive. Thus to explain the pricing kernel puzzle one needs to relax at least one of the three assumptions: complete markets, risk averse expected utility and correct beliefs. This, in principal, leads to three routes for solving the pricing kernel puzzle. And the first task is to understand the resulting mechanisms that lead to an increasing relation between the pricing kernel and aggregate endowments. Secondly, one should expect that relaxing these assumptions can lead to any structure of the pricing kernel as long as it is continuous and satisfies some boundary behavior. To see this recall the well-known "anything goes" result of Sonnenschein, Debreu and Mantel as it was called by Mas-Colell et al. (1995). Allowing for any characteristics of the investors (utility functions, beliefs and initial endowments) besides being positive the equilibrium state prices can be anything. Similar results hold true in incomplete markets (Hens and Pilgrim, 2002). However, certainly not

every combination of investors' characteristics is realistic. For example the distribution of wealth of investors and or households has certain well known properties that restrict the generalizations one can realistically do (Hildenbrand, 1994). Also, it is not realistic to claim any degree of heterogeneity of investors' beliefs (Ziegler, 2007). And finally the generalization used to solve the pricing kernel puzzle should be consistent with the main properties of the observed market structure. It is therefore natural to formulate two strongly related questions:

1. What are the mechanisms that can lead to a (locally) increasing relation between the pricing kernel and aggregate endowments in a financial market equilibrium?
2. Can the increasing part of the pricing kernel be generated by realistic assumptions on the parameters of examples illustrating these mechanisms?

While the main branch of literature tries to answer the second question, we are also interested in the first one. For a good understanding of the pricing kernel, it is necessary to understand the mechanism how different deviations from the standard assumptions influence the pricing kernel.

Given the three main assumptions of the standard model, we are thus lead to explore three possible reasons for the pricing kernel puzzle: risk-seeking behaviour, incorrect beliefs and incomplete markets.

Risk-seeking behaviour refers to the agent's tendency to gamble. While risk aversion is a standard assumption in finance, there is considerable empirical evidence that agents might show risk aversion for some ranges of returns and risk-seeking behaviour for others (typical examples can be found in Kahneman and Tversky (1979)). Formally, risk-seeking behaviour is described by a partially convex utility function. This makes the portfolio optimization problem of the agent non-robust but the resulting pricing kernel can be increasing.

Incorrect beliefs refer to situations where the probabilities used in the pricing kernel do not coincide with the weights used by the agents for evaluation of the payoffs. Reasons for such settings are misestimation (of the agent and/or the statistician), heterogeneous beliefs, ambiguity aversion or behavioural biases such as distortion. In such models, a bump can be viewed as a difference between the belief that determines the equilibrium and the measured belief. A third possibility is incomplete markets. While the market spanned by the index and its options is usually assumed to be complete, background risk and market frictions can lead to situations where the agents face a portfolio optimization problem in incomplete markets. In such a setting, the pricing

kernel which is relevant for the agents is not necessarily unique and the allocation chosen by different agents correspond to different pricing kernels. In this way, incompleteness can lead to an increasing pricing kernel.

In the literature, the pricing kernel is mainly analyzed from the econometric viewpoint. Researchers have taken great interest in estimating the pricing kernel. One often-used approach relies on a model of a representative agent in which the pricing kernel is a parametric function of the aggregate endowment. Market data is then used to estimate the parameters. Two of numerous examples are Brown and Gibbons (1985) and Hansen and Singleton (1983). Both use a pricing kernel implied by a power utility. The latter one additionally uses consumption data for the estimation. Due to the parametric form, the pricing kernel is necessarily decreasing. Dittmar (2002) approximates the marginal utility function by a Taylor series expansion and restricts the form of the series by imposing decreasing absolute prudence on investor's preference. He finds that the nonlinear pricing kernel outperforms a pricing kernel implied by power utility. Moreover, in order to describe the data well, he needs a cubic pricing kernel, which is evidence for increasing parts in the pricing kernel. Another approach is based on the no-arbitrage principle. While the techniques of this method have become more and more sophisticated, the basic approach has remained the same. Along the lines of Breeden and Litzenberger (1978), option data is used to estimate the state-price density. Other methods are used to estimate the historical distribution which is seen as a proxy for the physical probabilities. Some examples are Jackwerth and Rubinstein (1996); Ait-Sahalia and Lo (1998); Jackwerth (2000); Ait-Sahalia and Lo (2000); Brown and Jackwerth (2001); Rosenberg and Engle (2002); Yatchew and Härdle (2006); Barone-Adesi et al. (2008); Barone-Adesi and Dall'O (2009). The most robust observation in that part of the literature is that the global behaviour of the pricing kernel is generally decreasing. Often, but not always, there is an interval usually in the area of zero return where the pricing kernel is increasing. Note that this area is highly relevant since most of the returns of the market portfolio are in between  $-5\%$  to  $+5\%$ .

In the papers mentioned above, one finds many hypotheses which are evoked to explain the pricing kernel puzzle. One hypothesis is mistakes in the estimation, such as a faulty estimation procedure for the state-price density or for the physical probabilities. Another hypothesis is bad quality of option data. Jackwerth (2000) studies such explanations and finds that they can not fully explain the puzzle. Many empirical studies (Rosenberg and Engle, 2002; Detlefsen et al., 2010; Golubev et al., 2008) state that the increasing pricing kernel is evidence for increasing marginal utility of the representative agent. Jackwerth (2000) and Ziegler (2007) check whether a Peso problem could explain the puzzle. Shefrin (2005) explains the puzzle using a model with heterogeneous beliefs. Ziegler (2007) tests this explanation in his model and

comes to the conclusion that the degree of pessimism needed is implausibly high. More recently, Dierkes (2009) and Polkovnichenko and Zhao (2010) analyzed the pricing kernel in a setup with distorted beliefs and Gollier (2010) uses ambiguity aversion to explain the puzzle. Chabi-Yo et al. (2008) shows that state-dependence can explain the puzzle. More precisely, they introduce latent state variables upon which then fundamental variables or preferences in turn might depend. A combination of state-dependent utilities and additional state variables is also used in Benzoni et al. (2011). State-dependent utilities also appear in Grith et al. (2011). Bakshi et al. (2010) consider a model with heterogeneity in beliefs about returns and short-selling.

Given the huge variety of all these efforts, we thought that it is time to give a simple unifying framework of a financial market in which all of these hypotheses can be analyzed and compared. While Ziegler (2007) already compares different explanations regarding heterogeneous beliefs, we also include behavioural explanations and consider the case of incomplete markets which has so far not been mentioned in the literature. Moreover, given all these attacks at the structure of the pricing kernel one might wonder whether actually besides being positive it can have any structure. As we show in our Theorem 2, this is certainly not true. Even if none of the standard assumptions holds we can still show that the pricing kernel is generally decreasing as a function of aggregate resources. That means that it is certainly higher for very small aggregate resources than for very large aggregate resources.

The paper is organized as follows. In Section 2, we introduce the model and we define our notation of a financial market equilibrium. Section 3 considers the case of risk-averse agents having true and common beliefs in a complete market. The boundary behaviour of the pricing kernel is analyzed in Section 4. Section 5 is devoted to the study of the case of partially risk-seeking agents. Section 6 provides a detailed exposition of the case that risk-averse agents have incorrect beliefs. In Section 7, we look more closely at the pricing kernel in incomplete markets. Finally, Section 8 concludes the main results. In an effort to keep clear the main lines of the argument, some of the drier mathematical calculations are placed in appendices. For standard results in financial economics, the corresponding results in Magill and Quinzii (1996) are cited as one possible reference.

## 2 Setup

We consider a one-period exchange economy. Since the pricing kernel is defined as the ratio of the state prices and the beliefs of a given period all assumptions needed to model the pricing kernel are assumptions on the characteristics of that one-period economy. Generalizing the economy to

multiple periods cannot circumvent this fact but multi-period models might be used to motivate some of the assumptions made in the one-period model. Let  $\Omega = \{1, \dots, S\}$ ,  $S < \infty$  denote the states of nature in the second period. The set  $\mathcal{F} = 2^\Omega$  is the power algebra on  $\Omega$ , i.e., the set of all possible events arising from  $\Omega$ . Uncertainty is modeled by the probability space  $(\Omega, \mathcal{F}, P)$ , where the objective probability measure  $P$  on  $\Omega$  satisfies  $p_s = P[\{s\}] > 0$  for all  $s = 1, \dots, S$ , i.e., every state of the world has strictly positive probability to occur.

There are  $K+1$  assets, whose payoffs at date  $t = 1$  are described by  $A_k \in \mathbb{R}^S$ . The asset 0 is the risk-free asset with payoff  $A_0 = 1$ . The price of the  $k$ -th asset at date  $t = 0$  is denoted by  $q_k$ . The risk-free asset supply is unlimited and the price  $q_0$  is exogenously given by 1. This assumption does not restrict the generality of the model as we always may choose the bond as numeraire. In other words, the payoffs are already discounted. The prices of the other assets are endogenously derived by demand and supply. The market subspace  $\mathcal{X}$  is the span of  $(A_k)_{k=0,1,\dots,K}$ . Without loss of generality, we assume that no asset is redundant, i.e.,  $\dim(\mathcal{X}) = K+1$ , where obviously  $K+1 \leq S$  holds. The market is called complete if  $K+1 = S$  holds.

We consider a finite set  $\mathcal{I}$  of investors. Agent  $i$  has a stochastic income  $W^i \in \mathbb{R}_+^S$  at date 1. This summarizes initial wealth and initial holdings in stocks. The variable  $\theta^i = (\theta_0^i, \dots, \theta_K^i) \in \mathbb{R}^{K+1}$  denotes the  $i$ -th agent's portfolio giving the number of units of each of the  $K+1$  securities purchased (if  $\theta_k^i > 0$ ) or sold (if  $\theta_k^i < 0$ ) by agent  $i$ . Buying and selling these  $K+1$  securities is the only trading opportunity available to agent  $i$ . Thus, given the available securities, investor  $i$  can attain any payoff  $X = W^i + \sum_{k=0}^K A_k \theta_k^i$ , where  $\theta^i$  satisfies the budget restriction  $\sum_{k=0}^K q_k \theta_k^i \leq 0$ . Moreover, we assume that the resulting income must be non-negative in all states of nature, i.e.,  $X \geq 0$ . The subset of payoffs in  $\mathcal{X}$  that are positive and budget feasible for investor  $i$  is denoted by  $B^i(q)$ , i.e.,

$$B^i(q) := \left\{ X \in \mathbb{R}_+^S \mid X = W^i + \sum_{k=0}^K A_k \theta_k^i \text{ for } \theta^i \in \mathbb{R}^{K+1} \text{ s.t. } \sum_{k=0}^K q_k \theta_k^i \leq 0 \right\}.$$

The preferences of agent  $i$  are described by an increasing functional  $V^i : \mathcal{X} \rightarrow \mathbb{R}$ . This functional summarizes the utility function as well as the beliefs of the agents. We will explicitly define functionals in the next sections. In order to optimize the preference functional, agents may want to buy and sell assets. An allocation  $(X^i)_{i \in \mathcal{I}}$  is called feasible if the resulting total demand matches the overall supply. Formally, this means that the market-clearing condition

$$\sum_{i \in \mathcal{I}} \theta^i = 0$$

has to be satisfied. Note that the market-clearing conditions for the financial

contracts imply that the allocation  $(X_i)_{i \in \mathcal{I}}$  satisfies

$$\sum_{i \in \mathcal{I}} X^i = \sum_{i \in \mathcal{I}} W^i.$$

In a financial market equilibrium, prices of the assets are derived in such a way that the requested profiles  $X^i$  form a feasible allocation.

**Definition.** A price vector  $\hat{q} = (1, \hat{q}_1, \dots, \hat{q}_K)$  together with a feasible allocation  $(\hat{X}^i)_{i \in \mathcal{I}}$  is called a financial market equilibrium if each  $\hat{X}^i$  maximizes the functional  $V^i$  subject to  $B^i(\hat{q})$ .

Since the preference functional is strictly increasing, the agents would exploit arbitrage opportunities in the sense of a sure gain without any risk. This means that if there were such an opportunity, every agent would rush to exploit it and so competition will make it disappear very quickly. Thus, we conclude that the condition

$$\left\{ X \in \mathbb{R}_+^S \mid X = \sum_{k=0}^K A_k \theta_k^i \text{ for } \theta^i \in \mathbb{R}^{K+1} \text{ s.t. } \sum_{k=0}^K q_k \theta_k^i \leq 0 \right\} = \{0\}$$

is satisfied in equilibrium. This implies (Magill and Quinzii, 1996, Theorem 9.3) the existence of strictly positive Arrow security prices  $\pi = (\pi_1, \dots, \pi_S)$  summing to 1 such that  $q_k = \pi A_k$  holds for all assets  $k$ . Each set of such Arrow security prices then defines a pricing kernel

$$\frac{\pi}{p} := \left( \frac{\pi_1}{p_1}, \dots, \frac{\pi_S}{p_S} \right).$$

Note that the pricing kernel is not unique if the market is incomplete (Magill and Quinzii, 1996, Theorem 10.6). The Arrow security prices are also called the state-price density, the risk-neutral probabilities or the equilibrium price measure.

### 3 The Pricing Kernel Puzzle

In general, a puzzle is an observation that seems to contradict standard theory. In this section, we first explain what is meant by standard theory and we will give a short overview over the main findings of the empirical literature.

In the main part of the finance literature, agents are assumed to be risk-averse and to have common and true beliefs. Formally, the preference functional  $V^i$  of agent  $i$  is then given by

$$V^i(X) := E[U^i(X)] = \sum_{s=1}^S p_s U^i(X_s)$$



for a strictly increasing, strictly concave and differentiable utility function  $U^i : \mathbb{R}_+ \rightarrow \mathbb{R}$  satisfying the Inada-conditions

$$U^{i'}(0) := \lim_{x \searrow 0} U^{i'}(x) = +\infty,$$

$$U^{i'}(\infty) := \lim_{x \rightarrow \infty} U^{i'}(x) = 0.$$

Markets are often assumed to be complete. Under these assumptions, it follows that there is a decreasing relation between the pricing kernel and aggregate resources.

**Theorem 1.** *Consider a financial market satisfying  $\dim(\mathcal{X}) = S$  and let the preference functionals  $V^i$  be given as above. If  $(\hat{q}, (X^i)_{i \in \mathcal{I}})$  is a financial market equilibrium with pricing kernel  $\frac{\pi}{p}$ , then there exists a strictly decreasing function  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , such that*

$$\frac{\pi_s}{p_s} = f(W_s), \quad s = 1, \dots, S.$$

A formal proof is given in Magill and Quinzii (1996), Theorem 16.7. Intuitively, every agent forms his portfolio according to the first-order conditions, i.e., the requested profile has the form

$$\hat{X}_s^i = \left( U^{i'} \right)^{-1} \left( \lambda^i \frac{\pi_s}{p_s} \right)$$

for a suitable Lagrange parameter  $\lambda^i$ . Because of the decreasing marginal rate of substitution, this profile is a decreasing function of the pricing kernel. The same holds true for the sum of all profiles of the agents. Due to the market-clearing condition, this sum is equal to the aggregate resources. This implies that in the equilibrium the aggregate resources are a decreasing function of the pricing kernel. To understand these results more deeply we make a few remarks:

**Remark 1.** An equivalent way of demonstrating Theorem 1 goes via aggregation. Since markets are complete, equilibrium allocations are Pareto-efficient and can therefore be supported by the maximization of an aggregate utility, which as a positive sum of the individual utilities inherits concavity. Thus the pricing kernel being proportional to the gradient of the aggregate utility is a decreasing function of aggregate resources. For details see Chapter 16 in Magill and Quinzii (1996).

**Remark 2.** The assumptions of Theorem 1 can be relaxed. It is enough to assume that the utility functions  $U^i$  are increasing and concave (i.e., not necessarily strictly concave and not necessarily satisfying the Inada-conditions). Indeed, Theorem 1 of Dybvig (1988) and its generalisation in Appendix A of Dybvig (1988) show that the allocation  $\hat{X}^i$  of agent  $i$  and

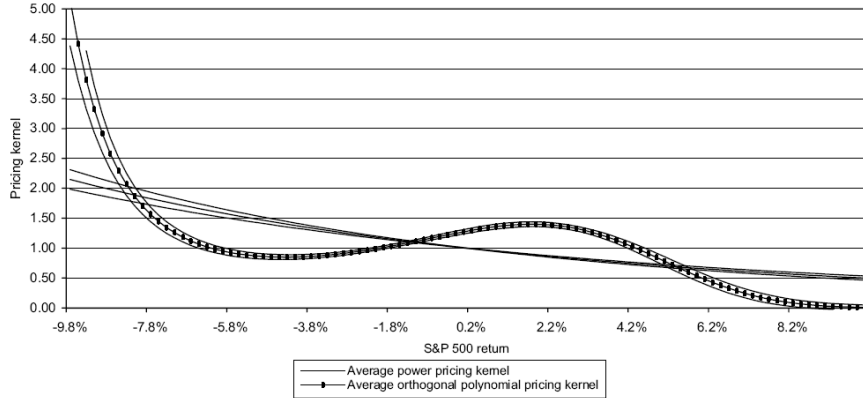


Figure 1: Rosenberg and Engle (2002), page 21<sup>2</sup>

the pricing kernel are anti-comonotonic<sup>1</sup>. Hence, this also holds for the sum over all agents. Using the market-clearing condition, it follows that the sum  $W = \sum_{i \in \mathcal{I}} \hat{X}^i$  and the pricing kernel are anti-comonotonic.

**Remark 3.** If we restrict ourselves to mean-variance type preferences, we end up in the CAPM which is the traditional example in finance. There, the pricing kernel is an affine decreasing function of the aggregate resources (Magill and Quinzii, 1996, Theorem 17.3).

Theorem 1 and Remark 3 presented a set of assumptions which implies a decreasing relation between the pricing kernel and aggregate resources. However, there is strong empirical evidence that this decreasing relation may be violated. This observation was made by several authors using different methods and different data sets (Jackwerth, 2000; Aït-Sahalia and Lo, 2000; Brown and Jackwerth, 2001; Rosenberg and Engle, 2002; Yatchew and Härdle, 2006). Furthermore, the estimated form of the pricing kernel is stable as well. Linear functions that fit the pricing kernel well, are decreasing. Using more flexible estimations, there is an interval usually in the area of zero returns where the pricing kernel is increasing. Note that in this area the return distribution has the mass of its observation. A typical form is presented in Figure 1. Thus we find it important to really understand the reasons for the shape of the pricing kernel. In the following section, we first explain the global behaviour of the pricing kernel. Later, we alternately skip one

<sup>1</sup>Comonotonicity of two random variables intuitively means that their realizations have the same rank order. In our setup, two random variables  $X^1$  and  $X^2$  are called comonotonic if  $(X_s^1 - X_{s'}^1)(X_s^2 - X_{s'}^2) \geq 0$  for all  $s, s' \in \{1, \dots, S\}$ . Random variables  $X^1$  and  $X^2$  are called anti-comonotonic if  $X^1$  and  $-X^2$  are comonotonic. See Föllmer and Schied (2004) for a general definition (Definition 4.76) and equivalent formulations (Lemma 4.83).

<sup>2</sup>We wish to thank Joshua Rosenberg and Robert Engle for providing this figure.

of three main assumptions complete markets, risk-aversion and correct beliefs and try to understand how the freedom gained may generate increasing pricing kernels.

## 4 Boundary behaviour of the pricing kernel

The preceding section showed that over the whole range (i.e. from the lowest to the highest aggregate endowment) the pricing kernel is monotonically decreasing if the market is complete, the agents are risk-averse and have correct beliefs. As we show later, as soon as we drop one of the assumptions, the statement does not hold anymore. The goal of this section is to show that however, even without those assumptions, the boundary behaviour (i.e. comparing the pricing kernel at the lowest and the highest aggregate endowment) is still decreasing, which means that affine functions fitting the pricing kernel well must be decreasing. More precisely, we show that if the aggregate endowment varies enough, the values of the pricing kernel for very low aggregate endowments is much higher than the values of the pricing kernel for very high aggregate endowments. In between, the pricing kernel may be increasing. Formally, the preferences of the agents are given by

$$V^i(X) := E_{P^i} [U^i(X)] = \sum_{s=1}^S p_s^i U^i(X_s).$$

The probability measure  $P^i$  represents the belief of the agent  $i$ . We assume the following boundary behavior of the utility functions: for sufficiently small and large values the utility function  $U^i : \mathbb{R}_+ \rightarrow \mathbb{R}$  of agent  $i$  is strictly increasing, concave and differentiable and satisfies the Inada condition, i.e.,

$$\begin{aligned} U^{i'}(0) &:= \lim_{x \searrow 0} U^{i'}(x) = +\infty, \\ U^{i'}(\infty) &:= \lim_{x \rightarrow \infty} U^{i'}(x) = 0. \end{aligned}$$

Observe that we do not assume that  $U^i$  is concave. Also, the market can be complete or incomplete. Recall that the aggregate endowment is denoted by

$$W_s = \sum_{i \in \mathcal{I}} W_s^i.$$

Without loss of generality, we assume that  $W_1 \leq W_2 \leq \dots \leq W_S$ . The next theorem states that for sufficiently large differences between aggregate endowments, the pricing kernel values at the extremes,  $W_1$  and  $W_S$ , are ordered in a decreasing way. This gives a foundation for the empirical observation that affine functions fitting the pricing kernel well, must be decreasing.

**Theorem 2.** *Suppose the utility functions of the agents satisfy the boundary behaviour. Then for every belief  $\tilde{p}_s > 0$  for all  $s = 1, \dots, S$  and every payoff matrix  $A$ , there is an aggregate endowment  $W \in \mathbb{R}^S$  such that in every financial market equilibrium a pricing kernel with respect to that belief is decreasing at the extremes, i.e.*

$$\frac{\pi_1}{\tilde{p}_1} \geq \frac{\pi_S}{\tilde{p}_S}.$$

*In particular, the state prices  $\pi$  can be chosen to be proportional to some agent  $i$ 's, marginal utility, i.e.,  $\pi_s = \lambda^i U^{i'}(X_s^i) p_s^i$ ,  $s = 1, \dots, S$ .*

The proof of these statements can be found in Appendix A. Less formally, Theorem 2 can be explained as follows: we fix the preferences of the agents, a belief  $\tilde{P}$  and a payoff matrix  $A$ . Then, we can find an aggregate endowment  $W$  such that every equilibrium in that economy supports at least one set of normalized state prices for which the pricing kernel for low values of aggregate endowments is much higher than the pricing kernel for high values. More precisely, the quotient exceeds  $\Delta$ . Moreover, the set of state prices is a reasonable one; there is at least one agent such that his utility gradient is proportional to the state prices.

The crucial assumption in the theorem is that the aggregate endowments varies enough, i.e. that there are states with very low aggregate endowments and states with very high aggregate endowments. In the case of a single agent, the first-order condition and Inada-conditions guarantee that the pricing kernel is high for very low aggregate endowment and low for very high aggregate endowment. If there are two or more agents, the allocations of the agents are small in the states with low aggregate endowments since all agents have positive allocations. Moreover, there is at least one agent who has a high allocation in the states where there is high aggregate endowments (due to the market clearing condition). For this agent, we can again look at the first-order condition and use the Inada-condition to deduce that the pricing kernel is generally decreasing. This theorem is powerful. It shows that even if there are locally risk-seeking agents having heterogeneous beliefs, the pricing kernel is generally decreasing if we consider sufficiently large variations in aggregate endowments.

## 5 S-shaped utility

In this section, we consider partially risk-seeking agents in a complete market ( $\dim(\mathcal{X}) = S$ ). More precisely, the agents have common and true beliefs, but they are not necessarily risk-averse. While risk aversion is a standard assumption in finance, there is considerable empirical evidence that agents might show risk aversion for some ranges of returns and risk-seeking behaviour for others (typical examples can be found in Kahneman and Tversky

(1979)). Formally, the preference functional is described by

$$V^i(X) = E[U^i(X)],$$

where  $U^i : \mathbb{R}_+ \rightarrow \mathbb{R}$  is strictly increasing. Hence, the main difference to the situation of Theorem 1 is that  $U^i$  is not necessarily concave. In the literature, the most prominent examples of non-concave utilities are the concave-convex-concave utility function suggested by Friedman and Savage (1948) and the convex-concave utility arising in Prospect theory (Kahneman and Tversky, 1979; Tversky and Kahneman, 1992).

Before we analyze whether risk-seeking behaviour is a possible reason for the findings in the empirical literature, we want to ensure that risk-seeking behaviour can, in principle, induce increasing pricing kernels. In Section 3 we argued that if the allocations of all agents are decreasing functions of the pricing kernel, also the aggregate endowment is a decreasing function of the aggregate endowment. So, in order to have a partially increasing pricing kernel, it is necessary that at least one agent has an allocation which is not a decreasing function of the pricing kernel.

**Example 1.** We consider an economy with two states, two assets and a single (representative) agent. The underlying probabilities are defined by  $p_1 = \frac{2}{3}$  and  $p_2 = \frac{1}{3}$ . The payoff matrix of the assets is given by

$$\mathbf{A} = \begin{pmatrix} 1 & \frac{2}{3} \\ 1 & 2 \end{pmatrix}.$$

The utility function of the agent is given by

$$U(x) = \begin{cases} (x-1)^{\frac{1}{3}}, & x \geq 1, \\ -(1-x)^{\frac{2}{3}}, & x < 1. \end{cases}$$

We can interpret 1 as his reference point. On the interval  $(0, 1)$ , the agent is risk-seeking and on the interval  $(1, \infty)$ , the agent is risk-averse. It is shown in Appendix B that  $X = (\frac{10}{9}, \frac{2}{3})$  is the optimal allocation of the agent for the prices  $\hat{q} = (1, 1)$ . In order to analyze the pricing kernel, we first determine the Arrow security prices defined by the equation  $(1, 1) = \pi A$ . It follows that  $\pi_1 = \frac{3}{4}$  and  $\pi_2 = \frac{1}{4}$ . Hence, the according unique pricing kernel is given by  $\frac{\pi}{p} = (\frac{9}{8}, \frac{3}{4})$ . We conclude that there is no decreasing relation between the pricing kernel and allocation of the single investor.

The same idea could be extended to construct an economy with two (or more) agents and trade. The example is presented in this form in order to make the economic principle as simple and transparent as possible. Figure 2 presents the indifference curves for Example 1. The quantities of the agent are measured in the usual way with the southwest corner as the origin. The

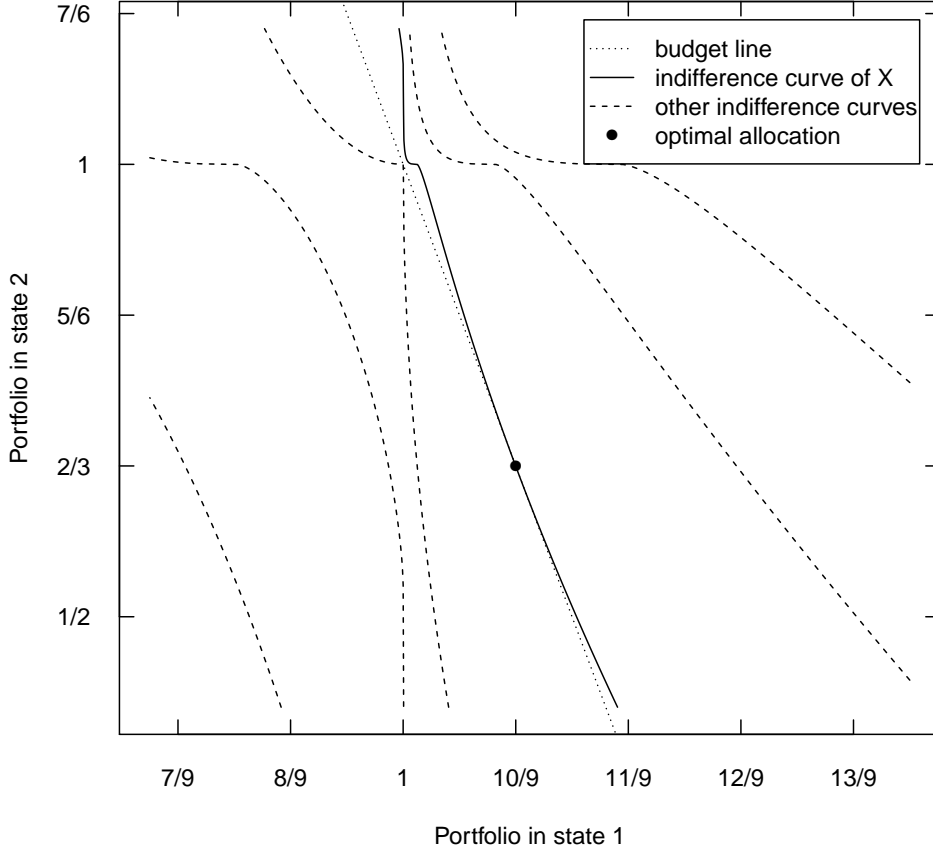


Figure 2: Indifference curves for an agent with a convex-concave utility function. The optimal allocation is an increasing function of the pricing kernel. The figure also shows the budget line and the indifference curves for the values  $-0.4$ ,  $-0.2$ ,  $0$ ,  $0.3$  and  $0.4$ .

allocation  $(\frac{10}{9}, \frac{2}{3})$  is optimal. The dotted line is the budget line for the agent. The solid line represents the indifference curve of the optimal allocation. The dashed lines represent indifference curves for the values  $-0.4$ ,  $-0.2$ ,  $0$ ,  $0.3$  and  $0.4$ . The indifference curves are not convex. In the area of the allocation  $(1, 1)$  the indifference curves are close to each other. This comes from the special form of the utility function around 1. Note that the optimal allocation and the budget line do not touch each other in  $(1, 1)$ , i.e.  $(1, 1)$  gives a strictly lower expected utility than  $(\frac{10}{9}, \frac{2}{3})$ . Moreover, one can see that the allocation  $(\frac{10}{9}, \frac{2}{3})$  lies on the convexification of the solid line although  $\frac{2}{3}$  is in the convex part of the utility function. This shows the difference between the convexification of the indifference curves and the concavification of the utility function. While an allocation without weight in the concavified<sup>3</sup> part

<sup>3</sup>The concavified part of a utility function  $U$  is  $\{U < U_c\}$  where  $U_c$  denotes the concave

leads to a decreasing relation between pricing kernel and allocation, lying on the convexification of the indifference curves is, as Example 1 shows, not sufficient for this.

Example 1 has also implications for related questions in portfolio optimization. In a similar setup, Dybvig (1988) shows that under assumptions on the utility function or the probabilistic structure, the optimal allocation and the pricing kernel are anti-comonotonic. This result is then generalized to more general setups and/or more general preferences, see, for instance, Carlier and Dana (2011) and He and Zhou (2011). Example 1 shows that one cannot drop all assumptions. For general probabilistic structures and general utility function, the optimal allocation and the pricing kernel are not necessarily anti-comonotonic.

While the above example and its implications are of theoretical interest, the non-robust allocation of the indifference curve already indicates that this may not be a reasonable explanation for the empirical findings. In order to substantiate this doubt, we next show that risk-seeking behaviour on the aggregate level (i.e. a representative agent with partially risk-seeking utility) can be excluded as an explanation for the pattern found in the economic literature. For this we use the next theorem proved in Appendix C.

**Theorem 3.** *Let  $U$  be an increasing, smooth utility function which is partially convex. Let  $C$  denote the interior of the interval where  $U$  is strictly convex. Moreover, let  $X^*$  be the optimal allocation for the pricing kernel  $\pi/p$  and initial endowment  $W$ . Then, there exists at most one state  $s \in \{1, \dots, S\}$  with  $X_s^* \in C$ .*

Before combining the theorem and the puzzle, we give the intuition for the theorem: partially convex utility functions induce risk-seeking behaviour because one wants to end up on the concave hull. One branch of literature (Bailey et al., 1980; Hartley and Farrell, 2002; Levy and Levy, 2002) uses this idea to argue that convex parts in the utility function do not influence the utility maximization problem. Due to the discrete structure in our setup, it is not possible to use this argument. However, the above theorem shows that it is optimal to allocate in such a way that at most one state lies in the area where  $U$  is strictly convex. In the case that there are a lot of states, the influence of the single state becomes small.

We now combine Theorem 3 and Figure 1 which then shows that a pricing kernel as in Figure 1 can not be explained by a representative agent having a partially risk-seeking utility function in a complete market economy. The utility function implied in the pricing kernel (formally by integrating the pricing kernel) in Figure 1 is concave-convex-concave. In particular, the utility function is strictly convex in the area between  $-3.5\%$  and  $2\%$ . However, a representative agent with such a utility function would, as shown in

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hull of  $U$ .

Theorem 3, avoid states between  $-3.5\%$  and  $2\%$ . But these states has to be hold by someone. Hence, prices need to adjust such that the pricing kernel is decreasing and it becomes more attractive again to hold the assets which provide payoffs in those states. This shows that the pricing kernel shown in Figure 1 is not consistent with a partially risk-seeking representative agent. We conclude that risk-seeking agents can, in principle, be seen as a possible argument for increasing areas in the pricing kernel. However, it only works for pathological examples with few states and, in our setup, it is not an explanation for the pattern found in the empirical literature. One other way of generating a decreasing utility gradient with a bump is to generalize the utility function to be state-dependent, i.e.,

$$V(X) := \sum_{s=1}^S p_s U_s(X_s)$$

where the utility  $U_s$  is concave for every  $s = 1, \dots, S$ . Then for any positive pricing kernel  $\pi_s/p_s$ ,  $s = 1, \dots, S$  there exist concave utility functions  $U_s$ ,  $s = 1, \dots, S$  such that the first-order condition

$$\frac{\pi_s}{p_s} = \lambda U'_s(X_s), s = 1, \dots, S$$

holds. In particular, no robustness problem arises. Assuming for example that the degree of risk-aversion is state-dependent, i.e.,

$$U_s(x_s) = \frac{x_s^{1-\alpha_s}}{1-\alpha_s}, s = 1, \dots, S,$$

one can generate the typical form of the pricing kernel by assuming that for small losses the investors are less risk-averse than for large gains or losses. Since state-dependent utilities are so flexible, the challenge then is then to give plausible restrictions such that the pricing kernel is generally decreasing and may have increasing areas. Attempts in this direction are done in Chabi-Yo et al. (2008) and Benzoni et al. (2011) by generalizing the model to multiple periods.

## 6 Incorrect beliefs

Let us now extend the discussion to situations where agents do not necessarily have common and true beliefs. In such a setting, the risky probabilities used in the pricing kernel do not coincide with the weights used by the agents for evaluation of the payoffs. In order to be able to analyze the shape of the pricing kernel and the robustness under aggregation, we need to separate different sources for incorrect beliefs. In the literature on the pricing kernel puzzle, heterogeneous beliefs and misestimation are considered as possible



explanations. These explanations amount to a conceptual problem in measurement: by which  $p$  do we define the pricing kernel? Considering agents with heterogeneous beliefs, this question is of fundamental importance; one has to carefully deal with aggregation of beliefs. On the other hand, analyzing the estimation procedure leads to practical problems arising in the measurement of the pricing kernel, see for instance Ziegler (2007). We suggest to additionally consider distortions, which amounts to a bias in decision making. We first consider the utility maximization problem for general incorrect beliefs and we then analyze the three phenomena independently.

In order to formally describe incorrect beliefs, we assume that the preferences of agent  $i \in \mathcal{I}$  are described by the functional

$$V^i(X) := E_{P^i} [U^i(X)] = \sum_{s=1}^S p_s^i U^i(X_s),$$

where  $P^i$  is a set function on  $(\Omega, \mathcal{F})$ . The set function  $P^i$  represents the subjective expectations of agent  $i$  about the future. For simplicity, we consider again the case of strictly concave utility functions satisfying the Inada conditions. In equilibrium, the agents maximize their utility functional  $V^i$  over the budget set  $B^i(\hat{q})$ . In particular, the payoffs are evaluated with respect to their own belief  $P^i$ . Formally, the allocation  $\hat{X}^i$  of agent  $i$  solves the maximization problem

$$\text{maximize } E_{P^i} [U^i(X)] \text{ subject to } X \in B^i(\hat{q}).$$

Using the relation  $\hat{q} = \pi A$ , the constraint  $\sum_{k=0}^K q_k \theta_k^i \leq 0$  can be rewritten (for details see Magill and Quinzii (1996), Page 83f) as  $\sum_{s=1}^S \pi_s X_s \leq \sum_{s=1}^S \pi_s W_s^i$ . The modification allows the Lagrange method to be used and we conclude that the allocation of agent  $i$  has the form

$$\hat{X}_s^i = \left( U^{i'} \right)^{-1} \left( \lambda^i \frac{\pi_s}{p_s^i} \right)$$

for a suitable Lagrange parameter  $\lambda^i$ . Note that the “true” real-world probability does not appear in the allocation of agent  $i$ . The agent invests proportional to the ratio of the price of the corresponding Arrow security and the subjective probability  $P^i$  of its success. Hence, the allocation  $\hat{X}^i$  of agent  $i$  is a decreasing function of the pricing kernel  $\frac{\pi}{p^i}$  with respect to his subjective probability measure. However, it is not necessarily a decreasing function of the pricing kernel  $\frac{\pi}{p}$  with respect to the “true” probability measure. Thus, in a model with a single (representative) agent, a bump can be viewed as a difference between his subjective probability measure and the estimated probability measure. In order to relate incorrect beliefs and the pattern found in the empirical literature we consider different sources of incorrect beliefs.

## 6.1 Distorted beliefs

We start with distortions as a special case for incorrect beliefs. Kahneman and Tversky (1979) show that agents tend to overweight extreme events. The simplest way to incorporate such a behaviour is to distort the given true probabilities of the states with an increasing, concave-convex function  $T : [0, 1] \rightarrow [0, 1]$ . One widely used parametric specification of the weighting function is due to Kahneman and Tversky (1979):

$$T(x) = \frac{x^\gamma}{(x^\gamma + (1-x)^\gamma)^{\frac{1}{\gamma}}}. \quad (4)$$

Another widely used specification is due to Prelec (1998):

$$T(P) = \exp(-(-\log(x))^\gamma). \quad (5)$$

Experimental studies typically find  $\gamma \in [0.5, 0.7]$ . The agent then evaluates the payoff with respect to the distorted probability  $T(p.)$ , i.e., the preference functional is given by

$$V(X) = \sum_{s=1}^S U(X_s)T(p_s).$$

Using the arguments above, we find that the allocation of agent  $i$  is

$$\hat{X}_s^i = (U')^{-1} \left( \lambda^i \frac{\pi_s}{T(p_s)} \right).$$

It is a decreasing function of  $\frac{\pi_s}{T(p_s)}$ , but it is not necessarily decreasing in  $\frac{\pi_s}{p_s}$ . This holds true for every agent and we conclude that the pricing kernel with respect to distorted probabilities is a decreasing function of the aggregate resources. However, in Figure 1 we plot  $\frac{\pi_s}{p_s}$  as a function of  $W_s$ . For a state with high probability (e.g., returns around zero in the case of S&P 500),  $T(p_s)$  is relatively underestimated. Hence,  $\frac{\pi_s}{T(p_s)}$  is relatively higher than  $\frac{\pi_s}{p_s}$ . For a state with low probability (e.g., extreme returns in the case of S&P 500),  $T(p_s)$  is relatively overestimated and  $\frac{\pi_s}{T(p_s)}$  is relatively lower than  $\frac{\pi_s}{p_s}$ . While the distortion applied on the probability weights is useful for illustration, it is limited in realistic applications with a lot of states. One possible extension is Rank-dependent expected utility (Quiggin, 1982; Yaari, 1987) where the cumulative distribution is distorted. In order to define the preference functional of a payoff  $X$ , we assume that the payoff is ordered in an increasing way. The preference functional is then given by

$$V(X) := \sum_{s=1}^S U(X_s)p_s T'(F_X(X_s)), s = 1, \dots, S, \quad (6)$$

where  $F_X$  denotes the probability distribution of  $X$ . In the original formulation, the preference functional is formulated in slightly different form. Under some assumptions, (6) then coincide with the original formulation and the first-order condition is

$$U'(X_s)p_sT'(F_X(X_s)) = \lambda\pi_s, \quad (7)$$

for details, see Polkovnichenko and Zhao (2010). In the literature, this relation is then used to deduce the probability distortions implied by the prices of the S&P500 index options. Dierkes (2009) assumes a parametric form for the utility and the distortion and calibrates (7) yielding parameters that are comparable to the one suggested in the experimental studies. In addition, Polkovnichenko and Zhao (2010) use (7) to estimate the distortion non-parametrically. They find that the distortion functions are time-varying. For some years the form is inverse  $S$ -shaped; for other years the form is slightly different. We apply the same approach on the estimates given in Figure 2 in Jackwerth (2000). The implied distortion is shown in Figure 3. For comparison, the same figure also shows the parametric form suggested in Prelec (1998) for  $\gamma = 0.7$ . While the form of the implied distortion is similar to the one suggested by experimental studies for low values, the form is substantially different for high values. The pricing kernel in the model with the implied distortion is decreasing. However, with the parametric form (4) or (5) the pricing kernel remains very bumpy. This shows that a fixed distortion is not sufficient to explain the estimates given in Jackwerth (2000). The reason for this, we believe, is that the estimate for the historical distribution is a bad proxy for probabilities perceived by the agent. We conclude that distortions, in isolation, is not a good explanation for the pattern found in the empirical literature and we consider a combination of misestimation and distortion in Section 6.3.

**Remark 4.** Recently, Gollier (2010) uses ambiguity aversion to explain the pricing kernel puzzle. In his model, the agent faces a finite set of beliefs and he assigns a probability to each of these beliefs. The pricing kernel is then defined with respect to a weighted belief. It is then shown that aversion to ambiguity affects the optimal allocation in a similar way as the distortion of probabilities in our framework, i.e. the probability that appears in the first-order condition is a transformation of the weighted probability distribution. Since the ambiguity for extreme returns is higher than for normal returns, the transformation for certain forms of ambiguity is similar to the form of the distortion in Prospect theory.

## 6.2 Misestimated and heterogeneous beliefs

In the analysis so far, we assumed that probability distribution estimated from historical data is a good proxy for the probability perceived by the

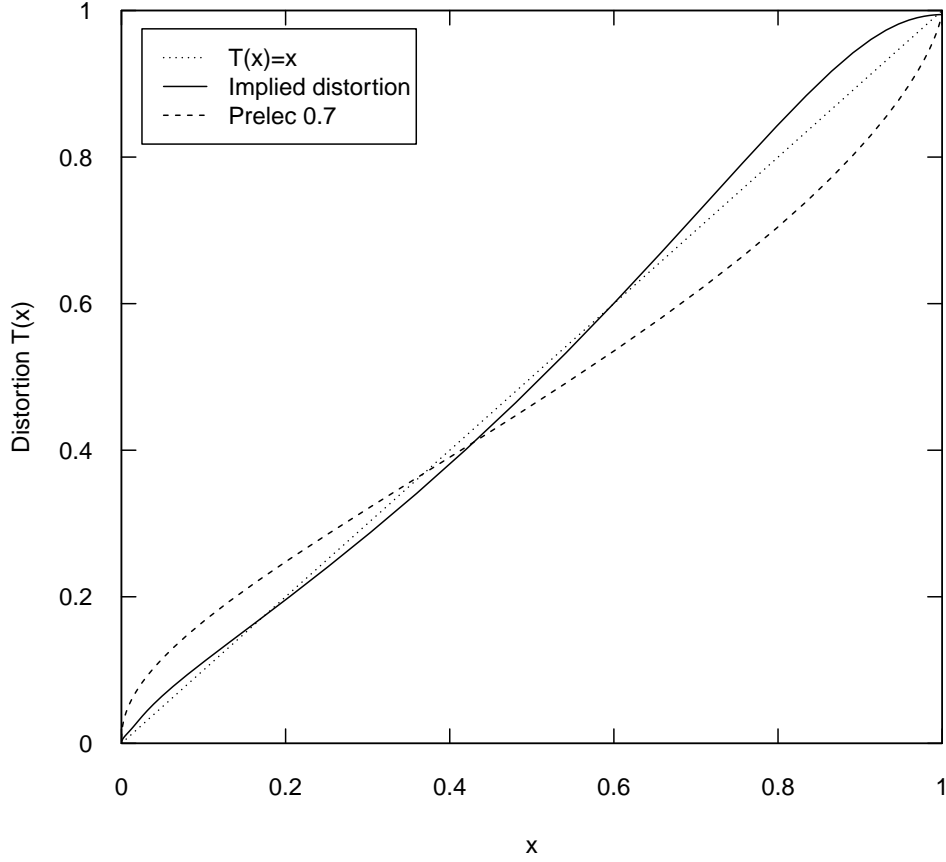


Figure 3: Distortion implied in data of Jackwerth (2000) and the parametric form of Prelec (1998) for  $\gamma = 0.7$ .

agents. However, this “past” probability can certainly differ from the representative beliefs  $\tilde{P}$  since the historical distribution is backward-looking whereas the beliefs of the agents are rather forward-looking. Following this line of arguments, the pattern found in the empirical literature might be only due to the misestimation of the beliefs. Assuming a fixed functional form for the utility of the agent (CRRA), the state-price density can be used to infer the belief. Ziegler (2007) tests this explanation and finds that the belief misestimation pattern that can be inferred is implausibly pessimistic.

In reality, different agents may have different views about the future, i.e., the beliefs  $P^i$  may differ. As argued above, the allocation of agent  $i$  is a decreasing function of  $\frac{\pi}{p^i}$ . On first view, it seems to be difficult to make precise statements about the pricing kernel. However, there are aggregation results for particular situations. One example is the CAPM with heterogeneous beliefs about the means. In such a situation, the aggregate belief can explicitly be derived (Gerber and Hens, 2006, Proposition 2.1). It takes into

account both the relative wealth and also the risk aversion of the agents. The wealthier and the less risk-averse agents determine the consensus belief more than the poor and more risk-averse agents. The pricing kernel with respect to the derived aggregate belief is then a linear decreasing function of the aggregate resources (Hens and Rieger, 2010, Section 4.4.1). Another prominent example can be found in (Shefrin, 2005, Theorem 14.1). It shows how agents with three sorts of heterogeneity (risk aversion, time discount factor and belief) aggregate into a single representative investor. The pricing kernel with respect to the belief of the representative agent is then a decreasing function of aggregate resources. In the case of arbitrary concave utility functions  $U^i$  and arbitrary beliefs  $P^i$ , it is possible (Calvet et al. (2001), Theorem 3.2 and Jouini and Napp (2007), Proposition 1) to define another common ‘consensus’ belief which, if held by all agents, would (after a possible reallocation of the initial endowments) generate the same equilibrium prices as in the actual heterogeneous world. The state prices in the ‘equivalent’ equilibrium remain the same because they just depend on the prices. Moreover, all the allocations in the equivalent equilibrium are decreasing functions of the pricing kernel with respect to the common ‘consensus’ belief. This transfers to the sum of all allocations which corresponds to the aggregate resources. So, even in the most general case, there is a belief such that the pricing kernel with respect to this belief is a decreasing function of the aggregate resources. A difference between the “true” probability and this belief is necessary for a partially increasing pricing kernel. Explicit expressions for the consensus belief are possible in explicit settings (Jouini and Napp, 2007; Ziegler, 2007). In a general model, Gollier (2007) analyzes how the consensus belief in a particular state depends on the degree of disagreement. Ziegler (2007) calibrates a heterogeneous-beliefs model with two or three groups of investors with lognormal beliefs to the semi-parametrically estimated state-price density of Aït-Sahalia and Lo (2000). Two groups of agents closely reproduces the estimates. Three groups of agents yields an almost perfect fit. However, the degree of pessimism required to explain the pattern seems to be implausibly large. We performed the same calibration with estimates in Jackwerth (2000) and the result are very similar<sup>4</sup>.

### 6.3 Combination of distortion and misestimation

In the previous sections we showed that distortion as well as misestimated and heterogeneous beliefs are in isolation not sufficient to explain the puzzle. We now combine the effects. More precisely, we assume that there is a representative agent. Similar to the framework of Ziegler (2007), the belief of the agent is given by a lognormal distribution. In addition, the agent distort the perceived distribution in the way described in Polkovnichenko and Zhao

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<sup>4</sup>For brevity we do not report detailed pictures of these estimates but they are available upon request.

(2010). The perceived density is then given by  $f_P(x) \cdot T'(F_P(x))$ , where  $f_P$  and  $F_P$  denotes the density function and the cumulative distribution function of the lognormal distribution. Combining (a discrete version of) the perceived density and the first-order condition given in (7) gives

$$U'(X_s)p_sT'(F_P(X_s)) = \lambda\pi_s, \text{ for } s = 1, \dots, S. \quad (8)$$

For the state-price density estimated in Jackwerth (2000) and a parametric form for  $U$  and  $T$ , the pricing kernel for a belief  $p_s$  is decreasing if (8) is satisfied. Allowing for any belief, this is certainly possible. However, it seems to be realistic to impose some restrictions. Following the model suggested by Ziegler (2007), we assume that the agents have log-normally distributed beliefs. Moreover, as well as in Ziegler (2007), we calibrate the implied state-price density rather than the implied pricing kernel. This ensures that the important part around return 1 has more weight than the tails. We assume that the utility is given by  $U(x) = \log(x)$ ; the distortion is given by the parametric form (5) suggested by Prelec (1998). This form has a simpler structure than (4) which has computational advantages. We then calibrate (8) to the estimate of the state-price density given Jackwerth (2000) using nonlinear least-squares yielding the parameters  $\lambda = 1.0177$ ,  $\mu = -0.0015$ ,  $\sigma = 0.0207$  and  $\gamma = 0.5401$  where  $(\mu, \sigma)$  are the parameters of the lognormally distributed belief of the agent. For comparison (8) is also calibrated without distortion (i.e.  $\gamma = 1$ ) yielding  $\lambda = 1.0398$ ,  $\mu = 0.0061$  and  $\sigma = 0.0341$ . The resulting densities are depicted in Figure 4.

The solid line is the state-price density estimated in Jackwerth (2000). The dotted line is the density obtained in the model without distortion and the dashed line is the density in the model with distortion. We see that the model with distortion - for contrast to the model without distortion - closely reproduce the empirical density. It captures the essential features well, in particular its thick left tail. Moreover, the parameter  $\gamma$  for the distortion is comparable to experimental studies. We conclude that a combination of distortion and misestimation can explain the pricing kernel puzzle.

## 7 Incomplete markets

Up to now, we restricted ourselves to the case of a complete market economy, i.e. we assumed that all consumption risk can be hedged by the financial markets, formally,  $\dim(\mathcal{X}) = S$ . In this section, we want to analyze the case  $\dim(\mathcal{X}) < S$ . To understand the impact of market incompleteness on the shape of the pricing kernel, we isolate this extension in the sense that we keep the other two assumptions: agents are risk-averse and have common and true beliefs.

Equilibrium asset prices  $\hat{q}$  are such that every agent maximizes the preference functional subject to his budget set and the market-clearing condition is

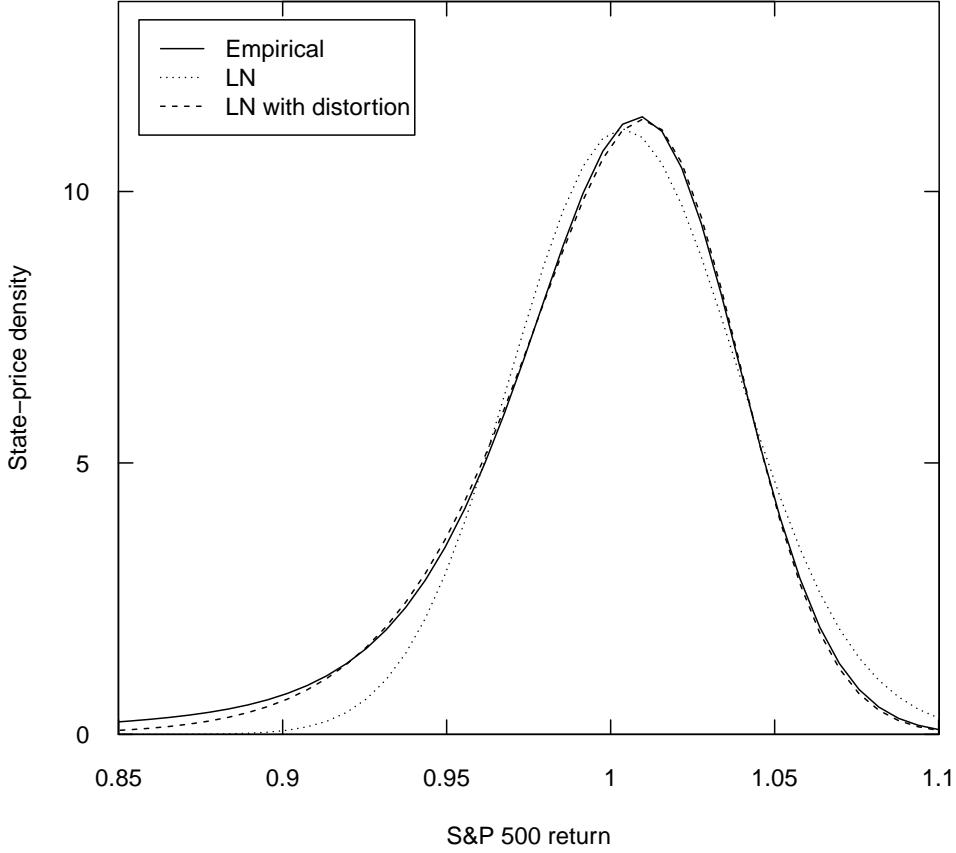


Figure 4: Fitting the state-price density with distorted beliefs closely reproduces the state-price density estimated in Jackwerth (2000). The density of the model without distortion, on the other hand, misses some essential features of the data.

satisfied. More technically, each agent solves a problem of the form

$$\max E[U^i(X)] \text{ over } X \in B^i(\hat{q}).$$

In incomplete markets there is a continuum of state prices  $\pi$  (of dimension  $S - \dim(\mathcal{X})$ ) satisfying the equation  $\hat{q} = \pi A$  and hence also a continuum of pricing kernels. The constraint  $\sum_{k=0}^K q_k \theta_k^i \leq 0$  can be rewritten using the pricing kernels and writing down the first-order conditions of that problem, it turns out (Magill and Quinzii, 1996, Theorem 12.4) that the solution has the same form as in the complete market case for a particular pricing kernel. More precisely, for every agent  $i$ , there is a state-price density  $\pi^i$  such that the allocation is of the form

$$\hat{X}^i = (U^{i'})^{-1} \left( \lambda^i \frac{\pi^i}{p} \right).$$

This shows that every allocation is a decreasing function of **some** pricing kernel. Hence, if there is a single representative agent, there exists some pricing kernel such that the allocation of the representative agent is a decreasing function of that pricing kernel. For heterogeneous agents, the situation is more involved. The allocation of each agent is a decreasing function of some pricing kernel  $\frac{\pi^i}{p}$  but the state-price density  $\pi^i$  “chosen” by the agents may differ. This can be used to give an example in which no pricing kernel is a decreasing function of aggregate resources. That is to say, in our example we have a unique equilibrium allocation and whatever state prices we select out of the continuum of state prices supporting it the resulting pricing kernel is not everywhere decreasing.

For an intuition, note that in incomplete markets the pricing kernel in one state does not only reflect the value of an additional unit of wealth in that state, but it also reflects the dependence between the states. Thus due to the incompleteness of markets the marginal utility in one state can have spill over effects into other states. We illustrate this phenomenon in the next example.

**Example 2.** We consider an economy with three states, two assets and two agents. The underlying probabilities are defined by  $p_1 = p_2 = p_3 = \frac{1}{3}$ . The payoff matrix of the assets is given by

$$\mathbf{A} = \begin{pmatrix} 1 & \frac{13}{15} \\ 1 & \frac{4}{3} \\ 1 & \frac{13}{15} \end{pmatrix}.$$

There are two agents. Both of them have utility  $U^1(x) = U^2(x) = \ln(x)$  and they have common and true beliefs, i.e., they evaluate utilities according to the probabilities  $p_1 = p_2 = p_3 = \frac{1}{3}$ . The initial endowment is given by  $W^1 = (\frac{76}{45}, \frac{19}{6}, \frac{212}{15})$  and  $W^2 = (\frac{418}{15}, \frac{33}{6}, \frac{382}{285})$ . It is shown in Appendix D that  $\hat{q} = (1, 1)$ ,  $\hat{X}^1 = (\frac{14}{9}, \frac{7}{2}, 14)$  and  $\hat{X}^2 = (28, \frac{7}{2}, \frac{28}{19})$  is a unique financial market equilibrium. In order to characterize the pricing kernels, we consider the state-price densities. Every collection of state-prices satisfies  $\pi_1 \frac{13}{15} + \pi_2 \frac{4}{3} + \pi_3 \frac{13}{15} = 1$  and it easily follows that all these probabilities can be written as a convex combination of the two extreme points  $(0, \frac{2}{7}, \frac{5}{7})$  and  $(\frac{5}{7}, \frac{2}{7}, 0)$ . We infer that  $\pi_2 < \max(\pi_1, \pi_3)$  holds for every state-price density. Because of  $p_1 = p_2 = p_3$ , the same holds true for the pricing kernel, i.e.,  $\frac{\pi_2}{p_2} \leq \max(\frac{\pi_1}{p_1}, \frac{\pi_3}{p_3})$ . The market portfolio  $W = X^1 + X^2 = (\frac{266}{9}, 7, \frac{294}{19})$  has the lowest value in state 2. We infer that **no** pricing kernel is a decreasing function of the aggregate resources.

Let us interpret this example. According to the marginal rate of substitution, agent 1 would like to transfer wealth from state 3 to state 1 and agent 2 would like to transfer wealth from state 1 to state 3. However, there is no asset (combination) which does this job. If one increases wealth in state 1, the wealth in state 3 automatically increases and vice versa. It is only



possible to transfer money from state 1 and state 3 to state 2 and vice versa. Both agents have high wealth in one state and relatively low wealth in the other two states. From a marginal-rate-of-substitution point of view, the low-wealth states are more important. Thus, both agents try to equalize the two states with the low value. More precisely, agent 1 would like to transfer money from state 2 to state 1 and agent 2 would like to transfer money of state 2 to state 3. This explains the low Arrow security price in state 2. But, the aggregate resources in state 2 are low. According to the marginal rate of substitution, an additional unit of wealth brings a high additional utility. We conclude that this information is, contrary to the case of complete markets, not contained in any pricing kernel. The pricing kernel reflects the dependence between the different states in the sense that both agents want to reduce their holding in state 2 relatively to the other states.

In order to decide whether incomplete markets is an explanation for the pattern found in the empirical literature, recall that the pricing kernel measured in the empirical literature is the pricing kernel projected onto index return states. Index option prices are used for the estimation. In particular at-the-money, there is a large number of options written on the used index (usually S&P 500). This is the reason why the literature usually assumes that the market subspace spanned by the index is complete. However, a lack of options for extreme strike prices as well as heterogeneous background risk leads to incomplete market situation. In the remainder of this section, we separately consider the effect of illiquid options for extreme strikes and heterogeneous background risk on the pricing kernel.

### 7.1 Incompleteness due to a lack of options

While there is a large number of relatively liquid options at-the-money, there are much less options for extreme strike prices. This leads to much less complete markets for extreme index return states. However, the bump in the pricing kernel occurs around at-the-money and it is determined by the at-the-money options. To make this intuition precise assume that the payoff matrix  $A \in \mathbb{R}^{S \times (K+1)}$  has a full rank sub-matrix  $\hat{A} \in \mathbb{R}^{J \times J}$  corresponding to some “middle states”  $j, j+1, \dots, J$  while  $A_{sk} = 0$  for  $s < j$  and  $s > J$ . Since the agent’s utility functions are additively separable, the economy then decomposes into a complete market economy allocating the resources in the middle states and  $S-J$  isolated states in which the autarky allocation results. Applying Theorem 1 to the “middle economy” shows that the pricing kernel is decreasing for non-extreme returns. We conclude that increasing areas in the pricing kernel due to a lack of options for extreme strike prices is not an explanation for the pattern found in the empirical literature.

## 7.2 Incompleteness due to background risk

Another source for incompleteness is heterogeneous background risk (see Franke et al. (1998) and references therein). Even so the market spanned by the index and its options is complete, there are other risks for example in labour income or real estate wealth which cannot be insured completely by the agents. To illustrate this, we consider four states and three assets. The underlying probabilities are denoted by  $p_1, \dots, p_4$ . For simplicity, we assume that  $p_2 = p_3$ . The payoff matrix of the assets is given by

$$\mathbf{A} = \begin{pmatrix} 1 & a & 0 \\ 1 & b & d \\ 1 & b & d \\ 1 & c & e \end{pmatrix}$$

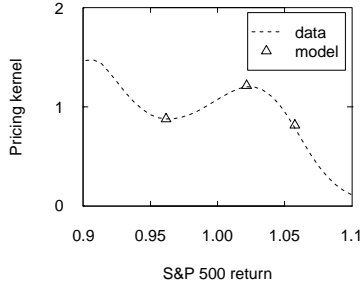
for values  $0 < a < 1 < b < c$  and parameters  $0 < d < e$  such that arbitrage is excluded. This market is incomplete and there are infinitely many sets  $(\pi_1, \dots, \pi_4)$  of state prices each of which can be written as a convex combination of two extreme points.  $\underline{\pi} = (\pi_1^*, \pi_{23}^*, 0, \pi_4^*)$  and  $\bar{\pi} = (\pi_1^*, 0, \pi_{23}^*, \pi_4^*)$ . However, given the information of the three assets we cannot separate state 2 and state 3. In this sense, the market spanned by the bank account, the “index” (asset 1) and the call option written on the index (asset 2) has only 3 observable states and the market which can be observed from the prices is complete. Hence there is a unique set of state prices for this market subspace which means that  $\pi_1$ ,  $\pi_2 + \pi_3$  and  $\pi_4$  are uniquely defined. But on the other hand, the individual agents may face risks which are not captured by the index. In our example this means that the individual endowments do not necessarily lie in the market subspace. For illustrative purpose, we consider the (extreme) case that the background risk only matters on individual level, i.e., the aggregate endowment lies again in the market subspace. In this way we end up in a situation that the index is a good proxy for the aggregate endowments; given the information of the assets the market is complete and admits a unique pricing kernel. However, the agents face a portfolio selection problem in the incomplete market.

In order to show that such a situation can lead to the typical patterns found in the empirical literature, we consider the simplest case of two log-agents facing opposites background risk. More precisely, we fix  $p_s$ ,  $\pi_s$  and the payoff matrix in such a way that no-arbitrage is satisfied (see Appendix E) and we can capture the essential features of the pricing kernel of Jackwerth (2000). As described above, the allocation in state  $s$  for agent  $i$  is of the form

$$\left(U^{i'}\right)^{-1} \left(\lambda^i \frac{\pi_s^i}{p_s}\right) \quad (9)$$

for some  $\lambda^i$  and a set  $\mu^i$  of state prices. For simplicity, we set  $\lambda := \lambda^1 = \lambda^2$  and  $\mu := \mu^1 = 1 - \mu^2$ . The parameters  $\lambda$  and  $\mu$  are determined in such a way

that the allocations (9) and the prices  $q = (1, 1, 1)$  form an equilibrium (see Appendix E). The corresponding values and the pricing kernel are shown in Figure 5. The pricing kernel has the typical increasing area. The constructed



(A) pricing kernel

$s$	$p_s$	$A_1$	$A_2$	$W^1$	$W^2$
1	0.4849	0.9616	0	0.4808	0.4808
2	0.1826	1.0217	0.975	0.2280	0.7937
3	0.1826	1.0217	0.975	0.7937	0.2280
4	0.15	1.0577	4.6466	0.5289	0.5289

(B) values

Figure 5: Panel (A) shows the pricing kernel for a complete market where risk-averse agents have heterogeneous background risk. For comparison, it also shows the pricing kernel implied in the data of Jackwerth (2000). Panel (B) shows the corresponding values for risky probabilities, the payoffs for assets  $A_1$  and  $A_2$  and the endowments  $W^1$  of agent 1 and  $W^2$  of agent 2. The endowments  $W^i$  are constructed via (9) for  $\mu = 0.7768$  and  $\lambda = 2.3227$ .

example has only three observable states, but the same idea can easily be extended to arbitrarily many states. Being so flexible, the challenge then is then to give plausible restrictions such that the pricing kernel is generally decreasing and may have increasing areas. One frequently used way to specify background risk on aggregate level in combination with state-dependent utilities is to introduce a second state variable (Garcia et al. (2003), Chabiyoyo et al. (2008) and Benzoni et al. (2011)). In those models, the pricing kernel does not only depend on the aggregate endowment but also on other state variables. Considering the pricing kernel as a function of aggregate endowment only, the other state variables can be seen as background risk which is not captured by the aggregate endowment.

## 8 Conclusion

In an economy with complete markets and risk-averse investors having correct beliefs the pricing kernel is a monotonically decreasing function of aggregate resources. In this paper we first show that neither of the three assumptions is needed for the pricing kernel to be generally decreasing, which means that affine functions fitting the pricing kernel, must be decreasing. However, as soon as we relax at least one assumption, one can construct examples where the pricing kernel has increasing parts. In this sense, the

pricing kernel puzzle results from a too-simplistic choice of the so-called “standard model”.

The explanation that (partially) risk-seeking agents induce an increasing area in the pricing kernel is non-robust and only works for pathological examples with few states. In a model with biased beliefs, the pricing kernel with respect to the consensus belief is a decreasing function of aggregate resources. Bumps correspond to a difference between measured “true” probability and the consensus probability. While the market spanned by the index and its options is usually assumed to be complete, background risk and market frictions can lead to situations where the agents face a portfolio optimization problem in incomplete markets. In incomplete markets, the pricing kernel also reflects information about the dependence between the different states. In this way, incompleteness can lead to an increasing pricing kernel in the complete subspace.

While all of these explanations in principle may lead to increasing pricing kernels, not all of them are able to explain this pattern with empirically realistic assumptions. Risk-seeking behaviour on the aggregate level can be excluded. Distorted, mistestimated or heterogeneous beliefs in isolation give unreasonable parameters. A combination of distorted and misestimated beliefs gives reasonable parameters. Also, the explanation based on incomplete markets seems to be realistic.

Having understood the pricing kernel (and its puzzle) in the one period model future research should focus on dynamic models in order to better understand how the pricing kernel changes e.g. with the direction of the market or with macro economic factors.

## A Proof of Theorem 2

*Proof.* We fix an arbitrary belief  $\tilde{P}$  and an arbitrary payoff matrix  $A \in \mathbb{R}^{S \times K}$ . We want to show that for any  $\Delta \geq 1$ , we can find  $W \in \mathbb{R}^S$  such that

$$\frac{\pi_1}{\tilde{p}_1} \geq \Delta \frac{\pi_S}{\tilde{p}_S}$$

holds. Let us fix  $W_1 > 0$ . Since the allocations have to be positive, it follows that  $X_1^i \leq W_1$  holds for all agents  $i$ . The strict monotonicity of  $U^i$  implies that  $\inf_{x \in [0, W_1]} U^{i'}(x) > 0$  is attained. If  $W_S \rightarrow \infty$ , the market clearing condition  $W_S = \sum_{i \in I} X_S^i$  implies that at least one agent (say  $i$ ) has an arbitrarily large allocation in state  $S$ . It follows from the Inada condition at  $\infty$ , that  $U^{i'}(X_S^i)$  is arbitrarily small. More precisely, we choose  $W_S$  large enough such that

$$\min_{i \in I} \frac{p_1^i \inf_{x \in [0, W_1]} U^{i'}(x)}{p_S^i U^{i'}\left(\frac{W_S}{|I|}\right)} \geq \frac{\tilde{p}_1}{\tilde{p}_S} \Delta,$$

where  $|I|$  denotes the number of agents. Then, it follows that at least one agent (say  $j$ ) has an allocation  $X_S^j \geq \frac{W_S}{|I|}$ . For agent  $j$ , we now use the Cass-Trick (Cass, 2006) to argue that

$$\frac{p_1^j U^{j'}(X_1^j)}{p_S^j U^{j'}(X_S^j)} = \frac{\pi_1^j}{\pi_S^j}. \quad (10)$$

Indeed, when all other agents' excess demand are in the span of the market subspace, then in equilibrium also agent  $j$  has an excess demand that satisfies the spanning constraint. Thus, we can let agent  $j$ 's budget restriction be unconstrained by the asset structure. Then, the first-order condition gives (10). Finally, we conclude

$$\frac{\pi_1^j}{\pi_S^j} = \frac{p_1^j U^{j'}(X_1^j)}{p_S^j U^{j'}(X_S^j)} \geq \min_{i \in I} \frac{p_1^i \inf_{x \in [0, W_1]} U^{i'}(x)}{p_S^i U^{i'}\left(\frac{W_S}{|I|}\right)} \geq \frac{\tilde{p}_1}{\tilde{p}_S} \Delta,$$

which is equivalent to  $\frac{\pi_1^i}{\tilde{p}_1} \geq \frac{\pi_S^i}{\tilde{p}_S} \Delta$ .  $\square$

## B Example 1

The state prices for  $\hat{q} = (1, 1)$  is  $\pi_1 = \frac{3}{4}$  and  $\pi_2 = \frac{1}{4}$ . In order to show that the allocation  $X = (\frac{10}{9}, \frac{2}{3})$  is optimal, we have to check that it solves

$$\max p_1 U(x_1) + p_2 U(x_2) \text{ subject to } \pi_1 x_1 + \pi_2 x_2 \leq \pi_1 \frac{10}{9} + \pi_2 \frac{2}{3} = 1.$$

Let us consider the cases  $x_1 = x_2 = 1$ ,  $x_1 > 1 > x_2$  and  $x_1 < 1 < x_2$  independently. In the case  $x_1 > 1 > x_2$ , plugging in the constraint, differentiating the term with respect to  $x_1$  and setting the resulting term equal to 0 give

$$\begin{aligned}\hat{x}_1 &= 1 + \left(\frac{p_1}{2p_2}\right)^3 \left(\frac{\pi_2}{\pi_1}\right)^2, \\ \hat{x}_2 &= 1 - \left(\frac{p_1}{2p_2}\right)^3 \left(\frac{\pi_2}{\pi_1}\right).\end{aligned}$$

Plugging the candidate  $\hat{x}_1$  into the second derivatives gives

$$-p_1 \frac{1}{3} \frac{2}{3} \left(\frac{p_1}{2p_2}\right)^{-5} \left(\frac{\pi_2}{\pi_1}\right)^{-\frac{10}{3}} + p_2 \frac{1}{3} \frac{2}{3} \left(\frac{\pi_2}{\pi_1}\right)^{-\frac{10}{3}} \left(\frac{p_1}{2p_2}\right)^{-4} < 0,$$

which shows that  $(\hat{x}_1, \hat{x}_2)$  is indeed a local maximum. The expected utility is

$$\frac{p_1^2 \pi_2^{\frac{2}{3}}}{4p_2 \pi_1^{\frac{2}{3}}}.$$

The same procedure for the case  $x_2 > 1 > x_1$  shows that

$$\begin{aligned}\hat{x}_1^* &= 1 - \left(\frac{\pi_1}{\pi_2}\right) \left(\frac{p_2}{2p_1}\right)^3, \\ \hat{x}_2^* &= 1 + \left(\frac{\pi_1}{\pi_2}\right)^2 \left(\frac{p_2}{2p_1}\right)^3\end{aligned}$$

is a local maximum. The expected utility is

$$\frac{p_2^2 \pi_1^{\frac{2}{3}}}{4p_1 \pi_2^{\frac{2}{3}}}.$$

Finally, comparing the local maxima and  $(0, 0)$  show that the allocation  $(\hat{x}_1, \hat{x}_2) = (\frac{10}{9}, \frac{2}{3})$  is optimal for  $p_1 = \frac{2}{3}$  and  $\pi_1 = \frac{3}{4}$ .

## C Proof of Theorem 3

*Proof.* By way of contradiction, we assume that there is an optimal allocation  $X$  with two states  $s$  and  $s'$  having values in  $C$ . We define

$$a := \pi_s X_s + \pi_{s'} X_{s'}$$

and consider the expression

$$f(x) := p_s U(x) + p_{s'} U\left(\frac{a - \pi_s x}{\pi_{s'}}\right).$$

Since  $U$  is convex, the same holds true for  $f$ . Maximizing a convex function gives a corner point solution. Thus, there is  $\tilde{x} \in C$  such that  $\frac{a - \pi_s \tilde{x}}{\pi_{s'}} \in C$  holds and  $f(\tilde{x}) > f(X_s)$  is satisfied. Let us define a new candidate  $\tilde{X}$  by

$$\begin{aligned}\tilde{X}_s &= \tilde{x}, \\ \tilde{X}_{s'} &= \frac{a - \pi_s \tilde{x}}{\pi_{s'}}, \\ \tilde{X}_{s''} &= X_{s''} \quad \text{for } s'' \in \{1, \dots, S\} \text{ and } s \neq s', s \neq s''.\end{aligned}$$

By construction,  $\tilde{X}$  is still affordable with initial endowment and gives a higher utility. This gives a contradiction to the optimality of  $X$ .  $\square$

## D Example 2

In order to show that  $\hat{q} = (1, 1)$ ,  $\hat{X}^1 = (\frac{14}{9}, \frac{7}{2}, 14)$  and  $\hat{X}^2 = (28, \frac{7}{2}, \frac{28}{19})$  is a financial market equilibrium, we have to check feasibility of the allocation  $(\hat{X}^i)_{i \in \mathcal{I}}$  and optimality of  $\hat{X}^i$  for the utility maximization problem of agent  $i$ . This can easily be done by verifying that the first order conditions for the agents' optimization problems are satisfied.

However since we also want to show the uniqueness of the equilibrium we have to dig into the setting a bit deeper. We first solve the utility maximization problem of the agents for prices  $\hat{q} = (1, 1)$  and we then check that the optimal allocation  $(\hat{X}^i)_{i \in \mathcal{I}}$  form a feasible allocation.

In order to maximize the expected utility, agent  $i$  chooses a strategy  $\theta^i = (\theta_0^i, \theta_1^i)^t$ , i.e., agent  $i$  buys  $\theta_j^i$  of asset  $j$  subject to his initial endowment. Formally, this can be described by the optimization problem

$$\begin{aligned}\text{maximize} \quad & \sum_{k=1}^3 p_k \ln(W_k^i + \theta_0^i + \theta_1^i A_{1k}) \\ \text{subject to} \quad & \theta_0^i + \theta_1^i q_1 \leq 0 \text{ and } W^i + \theta_0^i + \theta_1^i A_1 > 0\end{aligned}$$

for agent  $i$ . In the optimization problems, the initial endowment  $W^i$ , the price  $q_1$  and the probabilities are fixed. Due to the monotonicity of logarithm and the positivity of asset payoffs, we can replace the inequality in  $\theta_0^i + \theta_1^i q_1 \leq 0$  by equality and replace  $\theta_0^i$  by  $-\theta_1^i \cdot q_1$ . This simplifies the optimization problem to a maximization of a function depending on  $\theta_1^i$ . The boundary restriction  $W^i + \theta_0^i + \theta_1^i A_1 = W^i + \theta_1^i (A_1 - q_1) > 0$  has to be satisfied for every state  $s$ . The prices  $\hat{q} = (1, 1)$  exclude arbitrage and it follows that  $A_1 - q_1$  is both positive and negative for at least one state. We conclude that the boundary condition define a bounded interval of possible values for  $\theta_1^i$ . The property  $\ln(0) = -\infty$  implies that a candidate that satisfies  $W_k^i + \theta_0^i + \theta_1^i A_{1k} = 0$  in at least one coordinate, can not be optimal. Hence, a solution exists and satisfies the first-order conditions. Differentiating the

function  $\sum_{k=1}^3 p_k \ln (W_k^i + \theta_1^i (A_{1k} - q_1))$  with respect to  $\theta_1^i$  and setting the resulting term equal to 0 gives

$$\frac{A_{11} - q_1}{W_1^1 - q_1 \theta_1^1 + \theta_1^1 A_{11}} + \frac{A_{12} - q_1}{W_2^1 - q_1 \theta_1^1 + \theta_1^1 A_{12}} + \frac{A_{13} - q_1}{W_3^1 - q_1 \theta_1^1 + \theta_1^1 A_{13}} = 0 \quad (11)$$

for agent 1 and

$$\frac{A_{11} - q_1}{W_1^2 - q_1 \theta_1^2 + \theta_1^2 A_{11}} + \frac{A_{12} - q_1}{W_2^2 - q_1 \theta_1^2 + \theta_1^2 A_{12}} + \frac{A_{13} - q_1}{W_3^2 - q_1 \theta_1^2 + \theta_1^2 A_{13}} = 0 \quad (12)$$

for agent 2. Plugging in the explicit numbers for the price, the payoffs and the initial endowments and solving the equations for  $\theta_1^i$  show that 1 and  $\frac{649}{19}$  solve equation (11) and  $\frac{2648}{19}$  and  $-1$  solve equation (12). However,  $\theta_1^1 = \frac{649}{19}$  violates the boundary condition  $W^1 + \theta_1^1 (A_1 - q_1) > 0$  in state 1 and  $\theta_1^2 = \frac{2648}{19}$  violates the boundary condition  $W^2 + \theta_1^2 (A_1 - q_1) > 0$  in state 3. We conclude that  $\theta_1^1 = 1$  and  $\theta_1^2 = -1$  solve the utility maximization problems of the agents. In particular, we see that  $\sum_{i \in \mathcal{I}} \theta_1^i = 0$  holds, i.e., the market clearing condition is also satisfied.

In order to show uniqueness of the equilibrium, we solve equation (11) and (12) for a general price  $q_1$ . This gives again multiple solutions  $\theta_1^{1+}$  and  $\theta_1^{1-}$  for (11) and  $\theta_1^{2+}$  and  $\theta_1^{2-}$  for (12). Thus, there are four possible combinations and every combination determines an equilibrium price  $q_1$  via the market-clearing condition  $\sum_{i \in \mathcal{I}} \theta_1^i = 0$ :

- *Case +/+*: The +/+-combination gives the price  $q_1 = 1$ , which we already analyzed above.
- *Case -/+*: The market-clearing condition gives the price  $q_1 \approx 1.2461$ . It follows that  $\theta_1^1 \approx 23.07$ , and  $\theta_0^1 = -q_1 \theta_1^1 \approx -28.7527$ . This implies  $W^1 + \theta_0^1 + \theta_1^1 A_1 < 0$  in state 1, i.e., the boundary condition is violated. Hence, it cannot be an equilibrium.
- *Case +/-*: The market-clearing condition gives the price  $q_1 \approx 0.3412$ . It follows that  $\theta_1^1 \approx -18.99$ ,  $\theta_1^2 = -\theta_1^1$  and  $\theta_0^1 = -q_1 \theta_1^1 \approx 6.4834$ . This implies  $W^1 + \theta_0^1 + \theta_1^1 A_1$  in state 1, i.e., the boundary condition is violated. Hence, it cannot be an equilibrium.
- *Case -/-*: The market-clearing condition has no solution.

We conclude that only the +/+-combination leads to an equilibrium, which is the one we already analyzed above.

## E Calculations for the equilibrium in 7.2

We consider an economy with four states, three assets and two agents. The underlying probabilities are denoted by  $p_1, p_2, p_3$  and  $p_4$ . For simplicity, we



assume that  $p_2 = p_3 = \frac{p_{23}}{2}$  for some  $p_{23}$ . The payoff matrix of the assets is given by

$$\mathbf{A} = \begin{pmatrix} 1 & a & 0 \\ 1 & b & d \\ 1 & b & d \\ 1 & c & e \end{pmatrix}$$

for values  $a < 1 < b < c$  and parameters  $0 < d < e$  such that arbitrage is excluded. It can be shown that any state-price density can be written as convex combination of the two extreme points

$$\underline{\pi} = (\pi_1^*, \pi_{23}^*, 0, \pi_4^*) \text{ and } \bar{\pi} = (\pi_1^*, 0, \pi_{23}^*, \pi_4^*)$$

where

$$\begin{aligned} \pi_1^* &= \frac{be - cd - e + c - b + d}{ad - ae + be - cd}, \\ \pi_{23}^* &= \frac{e - ae + a - c}{ad - ae + be - cd}, \\ \pi_4^* &= \frac{-a + b - d + ad}{ad - ae + be - cd}. \end{aligned}$$

For simplicity, we choose

$$e := \frac{cp_1(-a + b - d + ad) - ap_4(-cd + c - b + d)}{ap_4(b - 1)}.$$

This implies

$$cp_1(-a + b - d + ad) = ap_4(be - cd - e + c - b + d) = ap_4(-cd + c - b + d) + eap_4(b - 1)$$

or putting differently

$$c = \frac{ap_4(be - cd - e + c - b + d)}{p_1(-a + b - d + ad)} = \frac{a\pi_1 p_4}{p_1 \pi_4}. \quad (13)$$

There are two agents. Both of them have utility  $U^1(x) = U^2(x) = \ln(x)$  and they have common and true beliefs. The endowment of agent 1 in state  $i$  is given by

$$W_i^1 = \left(U^{1'}\right)^{-1} \left( \lambda \frac{\mu \bar{\pi}_i + (1 - \mu) \underline{\pi}_i}{p_i} \right)$$

and the endowment of agent 2 in state  $i$  is given by

$$W_i^2 = \left(U^{2'}\right)^{-1} \left( \lambda \frac{(1 - \mu) \bar{\pi}_i + \mu \underline{\pi}_i}{p_i} \right)$$

for  $\lambda = \frac{2p_4}{c\pi_4^*}$  and

$$\mu = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{p_{23}c\pi_4^*}{b4p_4\pi_{23}^*}}.$$

By construction, this allocation is optimal for the given prices. Moreover, we claim that aggregate resources  $W^1 + W^2$  are equal to asset 1. In order to prove this, we start with state 4. It follows from the definition of  $\lambda$  that

$$\left(U^{1'}\right)^{-1} \left( \lambda \frac{\mu \bar{\pi}_4 + (1 - \mu) \underline{\pi}_4}{p_4} \right) + \left(U^{2'}\right)^{-1} \left( \lambda \frac{(1 - \mu) \bar{\pi}_4 + \mu \underline{\pi}_4}{p_4} \right) = \frac{2p_4}{\lambda \pi_4^*} = c.$$

For state 1, recall that it is shown in (13) that  $c = \frac{a\pi_1 p_4}{p_1 \pi_4^*}$ . This gives

$$\frac{2p_1}{\lambda \pi_1^*} = \frac{2p_1 c \pi_4^*}{2p_4 \pi_1^*} = \frac{2p_1 a \pi_1^* p_4 \pi_4^*}{2p_4 \pi_1^* p_1 \pi_4^*} = a.$$

For state 2 and 3, note first that

$$\mu - \mu^2 = \frac{p_{23} c \pi_4^*}{b 4 p_4 \pi_{23}^*} = \frac{p_{23}}{2 b \pi_{23}^* \lambda}.$$

Putting differently, we have

$$\frac{2b\lambda\pi_{23}^*}{p_{23}} = \frac{1}{\mu - \mu^2} = \frac{1}{\mu} + \frac{1}{1 - \mu},$$

which gives

$$b = \frac{\frac{p_{23}}{2}}{\lambda \mu q_{23}} + \frac{\frac{p_{23}}{2}}{\lambda (1 - \mu) q_{23}}.$$

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