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APPROXIMATE VERSUS EXACT EQUILIBRIA IN DYNAMIC ECONOMIES

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This paper develops theoretical foundations for an error analysis of approximate equilibria in dynamic stochastic general equilibrium models with heterogeneous agents and incomplete financial markets. While there are several algorithms that compute prices and allocations for which agents' first-order conditions are approximately satisfied ("approximate equilibria"), there are few results on how to interpret the errors in these candidate solutions and how to relate the computed allocations and prices to exact equilibrium allocations and prices. We give a simple example to illustrate that approximate equilibria might be very far from exact equilibria. We then interpret approximate equilibria as equilibria for close-by economies; that is, for economies with close-by individual endowments and preferences.

We present an error analysis for two models that are commonly used in applications, an overlapping generations (OLG) model with stochastic production and an asset pricing model with infinitely lived agents. We provide sufficient conditions that ensure that approximate equilibria are close to exact equilibria of close-by economies. Numerical examples illustrate the analysis.

KEYWORDS: Approximate equilibria, backward error analysis, perturbed economy, dynamic stochastic general equilibrium, computational economics.

1. INTRODUCTION

THE COMPUTATION OF EQUILIBRIA in dynamic stochastic general equilibrium models with heterogeneous agents is an important tool of analysis in finance, macroeconomics, and public finance. Many economic insights can be obtained by analyzing quantitative features of calibrated models. Prominent examples in the literature include, among others, Rios-Rull (1996) and Heaton and Lucas (1996).

Unfortunately there are often no theoretical foundations for algorithms that claim to compute competitive equilibria in models with incomplete markets or overlapping generations. In particular, since all computation suffers from truncation and rounding errors, it is obviously impossible to numerically verify that the optimality and market clearing conditions are satisfied, and that a competitive equilibrium is found. The fact that the equilibrium conditions are approximately satisfied generally does not yield any implications on how well the computed solution approximates an exact equilibrium. Computed allocations and prices could be arbitrarily far from competitive equilibrium allocations and prices.

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In this paper we develop an error analysis for the computation of competitive equilibria in models with heterogeneous agents where equilibrium prices are infinite dimensional. We define an ϵ-equilibrium as a collection of finite sets of choices and prices such that there exists a process of prices and choices that takes values exclusively in these sets and for which the relative errors in agents' Euler equations and the errors in market clearing conditions are below some small ϵ at all times. Existing algorithms for the computation of equilibria in dynamic models can be interpreted as computing ϵ-equilibria, and the finiteness of ϵ-equilibria allows us to computationally verify if a candidate solution constitutes an ϵ-equilibrium. To give an economic interpretation of the concept, we follow Postlewaite and Schmeidler's (1981) analysis for finite economies and interpret ϵ-equilibria as approximating exact equilibria of a close-by economy.

In finite economies the problem of interpreting ϵ-equilibria is easiest illustrated in a standard Arrow–Debreu exchange economy. Scarf (1967) proposes a method that approximates equilibria for any given finite economy in the following sense: Given individual endowments e^i for individuals $i = 1, \dots, I$ and an aggregate excess demand function $\xi(p, (e^i)_{i=1}^I)$, and given an $\epsilon > 0$, the method finds a $\bar{p}$ such that $\|\xi(\bar{p}, (e^i)_{i=1}^I)\| < \epsilon$. As Anderson (1986) points out, this fact does not imply that it is possible to find a $\tilde{p}$ such that $\|\tilde{p} - p^*\| < \epsilon$ for some exact equilibrium price vector p^*. Richter and Wong (1999) make a similar observation. They examine the problem of the computation of equilibria from the viewpoint of computable analysis and point out that while Scarf's algorithm generates a sequence of values converging to a competitive equilibrium, knowing any finite initial sequence might shed no light at all on the limit.

However, if individual endowments are interior and if the value of the excess demand function at $\bar{p}$, $\|\xi(\bar{p}, (e^i))\|$, is small, then $\bar{p}$ is an equilibrium price for a close-by economy. Homogeneity of aggregate excess demand implies that if $\bar{p} \cdot \xi(\bar{p}, (e^i)) = 0$, then $\|(\bar{p}, (e^i)) - (p^*, (\tilde{e}^i))\| < \epsilon$ with $\xi(p^*, (\tilde{e}^i)) = 0$. Figure 1 displays an equilibrium correspondence, which maps endowments into equilibrium prices. The computed price for the original economy is far away from the unique exact equilibrium price. No small perturbation of this price is an equilibrium price for the economy. However, there is an economy with close-by endowments for which the computed price is an equilibrium price.

Researchers rarely know the exact individual endowments of agents anyway, and if close-by specifications of exogenous variables lead to vastly different equilibria, it will be at least useful to know one possible equilibrium for one realistic specification of endowments. As Postlewaite and Schmeidler (1981) put it, "If we don't know the characteristics, but rather, we must estimate them, it is clearly too much to hope that the allocation would be Walrasian with respect to the estimated characteristics even if it were Walrasian with respect to the true characteristics."

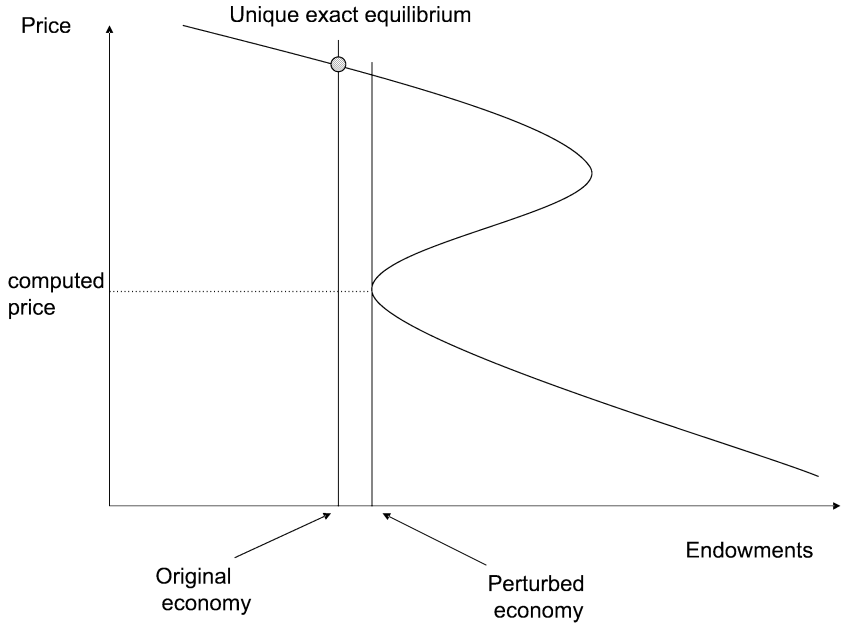

FIGURE 1.—So close and yet so far.

This issue has long been well understood from the viewpoint of computational mathematics. In general, sources of errors in computations can be classified into three categories. First, there are errors due to the theory: Economic models typically contain many idealizations and simplifications. Second, there are errors due to the specification of exogenous variables: The economic model depends on parameters that are themselves computed approximately and are the results of experimental measurements or the results of statistical procedures. Third, there are errors due to truncation and rounding: Each limiting process must be broken off at some finite stage, and because computers usually use floating point arithmetic, round-off errors result. There exists a debate within the applied economic literature (which uses computations) about the trade-off between the first and third sources of errors, but there is surprisingly little discussion about a possible trade-off between the second and third sources. This paper explores how this latter trade-off can be used to interpret approximate solutions to dynamic general equilibrium models via backward error analysis.

Backward error analysis is a standard tool in numerical analysis that was developed in the late 1950's and 1960's (see Wilkinson (1963) or Higham (1996)). Surprisingly, this tool has not been widely used in economics.[2] In backward

[2]Judd's textbook (1998), for example, mentions backward error analysis and provides a citation from the numerical analysis literature, but never applies the concept to an economic problem.

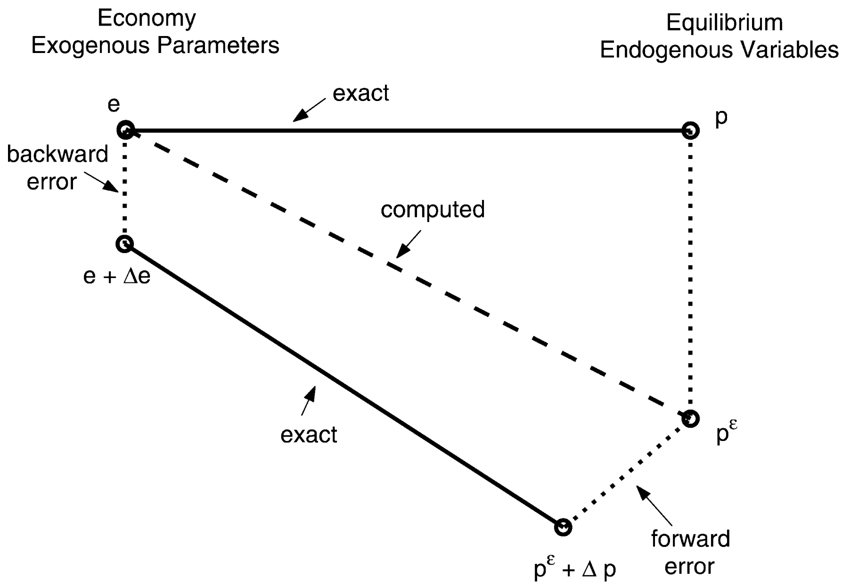

FIGURE 2.—Mixed forward–backward error analysis.

error analysis exogenous parameters are given, an approximate solution is computed, and then the necessary perturbations in exogenous parameters are determined for the computed solution to be exact. The focus of our analysis of popular models in Sections 5 and 6 of this paper is the calculation of backward errors. Due to the nature of economic problems we cannot perform "pure" backward error analysis and only perturb exogenous parameters. Instead, we compute bounds on perturbations of both exogenous parameters and endogenous equilibrium values. Higham (1996) calls this "mixed" forward–backward error analysis. Figure 2 elucidates this concept. Ideally we are interested in the exact equilibrium for the original economy and would like to provide a bound on the distance between the computed and the exact equilibrium. However, we argue that this may often be impossible and, instead, we interpret the computed equilibrium as a good approximation (small forward error in the figure) of the exact equilibrium of an economy that is a slight perturbation (backward error) of the original economy.

The analysis in our paper is, from a theoretical perspective, perhaps closest to Mailath, Postlewaite, and Samuelson's (2004) discussion of ϵ-equilibria in dynamic games. An important difference is that Mailath et al. allow for perturbations in the instantaneous payoff functions of the game. In our framework

Sims (1989) and Ingram (1990) use the terminology "backsolving" for a method for solving nonlinear, stochastic systems. This concept is fairly unrelated to backward error analysis.

this can lead to preferences over payoff streams that are far away from the original preferences. Therefore, we do not consider these.

For models with a single agent, Santos and his coauthors examine forward error bounds both on policy functions and on allocations (Santos and Vigo-Aguiar (1998), Santos (2000), and Santos and Peralta-Alva (2002)). They derive sufficient conditions under which it is possible to estimate error bounds from the primitive data of the model and from Euler equation residuals. However, most of these results do not generalize to models with heterogeneous agents and incomplete markets. No sufficient conditions are known that allow the derivation of error bounds on computed equilibrium prices and allocations in the models considered in this paper.

The paper is organized as follows. In Section 2 we illustrate the main intuition in a simple two-period example. Section 3 outlines an abstract dynamic model and defines what we mean by close-by economies. Section 4 develops the theoretical foundations of our method. In Section 5 we apply this method to a model with overlapping generations and production. In Section 6 we apply the methods to a version of Lucas' (1978) asset pricing model with heterogeneous agents.

2. THE MAIN INTUITION IN A TWO-PERIOD ECONOMY

In this section we demonstrate the main themes of this paper in a simple two-period model. We first show how competitive equilibria can be characterized by a system of equations that relates endogenous variables in one period to endogenous variables of the next period. These equations, which we refer to as the equilibrium equations, enable us later in the paper to describe infinite equilibria with finite sets. Second, we define an ϵ-equilibrium and provide an example that shows that ϵ-equilibrium prices and allocations can be a terrible approximation to exact equilibria. We show that, in the example, perturbations in individual endowments can rationalize ϵ-equilibria as exact equilibria.

EXAMPLE 1: Consider a simple pure exchange economy with two agents, two time periods, and no uncertainty. There is a single commodity in each period: agents' endowments are (e_0^i, e_1^i) for $i = 1, 2$. Agents can trade a bond that pays one unit in the second period; the price of the bond is denoted by q. Agents' bond holdings are θ^i, $i = 1, 2$. Agents preferences are represented by time-separable utility $U^i(x_0, x_1) = v_i(x_0) + u_i(x_1)$, $i = 1, 2$, for increasing, differentiable, and concave functions $v_i, u_i : \mathbb{R}_+ \to \mathbb{R}$.

A competitive equilibrium is a collection of choices $(c^i, \theta^i)_{i=1,2}$ and a bond price q such that both agents maximize utility and markets clear, i.e., $\theta^1 + \theta^2 = 0$ and for both $i = 1, 2$,

$$(c^i, \theta^i) \in \arg \max_{c\in\mathbb{R}_+^2, \theta\in\mathbb{R}} U^i(c) \quad \text{s.t.} \quad c_0 = e_0^i - q\theta, \quad c_1 = e_1^i + \theta.$$

To represent equilibria for infinite-horizon models we want to derive a system of equations that links endogenous variables (i.e., choices and prices) today to endogenous variables in the next period. In this simple example, we define the vector of relevant endogenous variables to consist of current consumption, current portfolios, and current prices, $z = ((c^i, \theta^i)_{i=1,2}, q)$. (Even though agents do not trade the bond in the second period we include zero bond holdings and a zero price in the state variable z_1 for that period. This setup has the advantage that the resulting equilibrium expressions look very similar to those in the infinite-horizon problems that we examine in the main part of the paper.)

In this two-period example, we define a system of equations $h(z_0, \kappa, z_1)$ such that $((\bar{c}^i, \bar{\theta}^i)_{i=1,2}, \bar{q}) \in \mathbb{R}^2_+ \times \mathbb{R}^2 \times \mathbb{R}^2_+ \times \mathbb{R}^2 \times \mathbb{R}^2_+$ is a competitive equilibrium if and only if there exists $\kappa = (\kappa^1, \kappa^2) \in \mathbb{R}^2_+ \times \mathbb{R}^2_+$ such that $h(\bar{z}_0, \kappa, \bar{z}_1) = 0$, with $\bar{z}_0 = ((\bar{c}^i_0, \bar{\theta}^i_0)_{i=1,2}, \bar{q}_0)$ and $\bar{z}_1 = ((\bar{c}^i_1, 0)_{i=1,2}, 0)$. The system is

$$h(z_0, \kappa, z_1) = \begin{cases} -q_0 v_i'(c_0^i) + u_i'(c_1^i) - q_0\kappa_0^i + \kappa_1^i & (i = 1, 2), \\ c_0^i - (e_0^i - q_0\theta_0^i) & (i = 1, 2), \\ c_1^i - (e_1^i + \theta_0^i) & (i = 1, 2), \\ \kappa_0^i c_0^i & (i = 1, 2), \\ \kappa_1^i c_1^i & (i = 1, 2), \\ \theta_0^1 + \theta_0^2. \end{cases}$$

An exact equilibrium is characterized by $h(\cdot) = 0$, but computational methods can rarely find exact solutions. All one can usually hope for is to find an ϵ-equilibrium, namely (z_0, z_1), such that $\min_{\kappa \in \mathbb{R}^4_+} \|h(z_0, \kappa, z_1)\| < \epsilon$. Unfortunately, even in this very simple framework, one can construct economies where ϵ-equilibria can be arbitrarily far from exact equilibria.

2.1. *Approximate Equilibria Can Be Far from Exact*

Consider the following class of economies parameterized by $\delta > 0$:

$$v_1(x) = x, \quad u_1(x) = -\frac{1}{x}, \quad e^1 = (2, \delta), \quad \text{and}$$

$$v_2(x) = -\frac{1}{x}, \quad u_2(x) = x, \quad e^2 = (0, 2).$$

We can easily verify that a competitive equilibrium is given by

$$q_0 = \frac{1}{(2+\delta)^2}, \quad \theta_0^1 = 2 = -\theta_0^2,$$

$$c^1 = \left(2 - \frac{2}{(2+\delta)^2}, 2 + \delta\right), \quad c^2 = \left(\frac{2}{(2+\delta)^2}, 0\right).$$

This equilibrium is unique for $\delta > 0$. In addition, for $\delta < \frac{1}{\sqrt{4-\epsilon}} - \frac{1}{2}$, the following values of the asset price, asset holdings, and consumption vectors yield an ϵ-equilibrium:

$$q_0 = 4, \quad \theta_0^1 = -\theta_0^2 = \frac{1}{2}, \quad c^1 = \left(0, \frac{1}{2} + \delta\right), \quad c^2 = \left(2, \frac{3}{2}\right).$$

All equations except for $h^1(\cdot) = 0$ for agent 1 hold with equality. The error in this equation is below ϵ by construction.

This example shows that for any (arbitrarily small) $\epsilon > 0$ we can construct an economy and an ϵ-equilibrium that is far from an exact equilibrium both in allocations and prices. Furthermore, it is worth noting that agents' welfare levels differ significantly between the exact equilibrium and the ϵ-equilibrium. For very small δ, utility levels in the exact equilibrium are approximately $(U^1, U^2) \approx (1, -2)$, while in the ϵ-equilibrium they are approximately $(U_\epsilon^1, U_\epsilon^2) \approx (-2, 1)$. No matter how one looks at it, the ϵ-equilibrium is evidently a terrible approximation for the exact equilibrium.[3] This observation motivates us to interpret ϵ-equilibria as approximate equilibria for close-by economies.

2.2. *Perturbing Endowments Makes Approximate Equilibria Exact*

In the example, we can easily explain the idea of mixed forward–backward error analysis and how an ϵ-equilibrium can be understood as approximating an exact equilibrium of a close-by economy. At the ϵ-equilibrium $q_0 = 4$, $\theta_0^1 = -\theta_0^2 = 1/2$, the only equilibrium equation that does not hold with equality is

$$h^1 = -q_0 + \frac{1}{(e_1^1 + \theta_0^1)^2} = -4 + \frac{1}{(e_1^1 + 1/2)^2} = 0.$$

If we replace the endowments e_1^1 by $\hat{e}_1^1 = e_1^1 + w$ for some small w we can evidently set $h^1 = 0$ by using $w = -\delta$. The corresponding perturbation of agent 1's consumption is $c_1^1 = c_1^1 - \delta$. The ϵ-equilibrium is exact for the perturbed economy. In this mixed forward–backward error analysis we perturbed the value of the endogenous variable c_1^1 and the value of the exogenous parameter e_1^1.

While this is the main idea underlying our error analysis, there is one additional complication that arises when agents live for many periods: Errors may propagate over time, and no sensible bounds on perturbations in endowments can be derived by perturbing endowments every period. In Section 6 we discuss this problem and a possible solution.

[3]For finite economies there do exist sufficient conditions that relate approximate equilibria to exact equilibria (see, for example, Blum et al. (1998, Chapter 8) and Anderson (1986)). However, these cannot be generalized to the infinite-horizon economies we consider in this paper.

3. A GENERAL MODEL

In this section we fix the main ideas in an abstract framework that encompasses both economies with overlapping generations and economies with infinitely lived agents, as well as economies with and without production. In Sections 5 and 6 we consider two standard models and show how to apply the methods developed in this and the next section.

3.1. *The Abstract Economy*

Time and uncertainty are represented by a countably infinite tree Σ. Each node of the tree, $\sigma \in \Sigma$, represents a date-event and can be associated with a finite history of exogenous shocks $\sigma = s^t = (s_0, s_1, \dots, s_t)$. The process of exogenous shocks (s_t) is a Markov chain with finite support $\mathcal{S} = \{1, \dots, S\}$. Given an $S \times S$ transition matrix Π, we define probabilities for each node by $\pi(s_0) = 1$ and $\pi(s^t) = \Pi(s_t | s_{t-1})\pi(s^{t-1})$ for all $t \geq 1$.

There are L commodities, $l \in \mathcal{L}$, at each node. There are countably many individuals $i \in \mathcal{I}$ and countably many firms $k \in K$. An individual $i \in \mathcal{I}$ is characterized by his consumption set X^i, his individual endowments $e^i \in X^i$, his preferences $P^i = \{(x, y) \in X^i \times X^i : x \succ^i y\} \subset X^i \times X^i$, and trading constraints. A firm $k \in K$ is characterized by its production set Y^k. With incomplete financial markets, the objective of the firm is in general not well defined, and there is a large, but inconclusive literature on the subject, following Drèze (1974). In the example below we circumvent the problem by assuming that firms are only active in spot markets. An economy $\mathcal{E}$ is characterized by a demographic structure, assets, technologies and preferences, endowments, and trading constraints. In the concrete models below we describe $\mathcal{E}$ explicitly.

The original economy is assumed to be Markovian. The number of agents active in markets at a given node is finite and time-invariant, but it may depend on the underlying exogenous shock. Agents maximize time and state-separable utility. Firms only make decisions on spot markets. All individual endowments, payoffs of assets, production sets of firms, and spot utility functions of individuals are time-invariant functions of the shock, s, alone.

3.1.1. *Perturbations and Backward Errors*

For a mixed forward–backward error analysis, we must specify which exogenous parameters can be perturbed and which kind of perturbations are permissible. The exact set of admissible perturbations will be governed by the economic application in mind. It will become clear below that one must always allow for perturbations at all nodes in the event tree and that the resulting economy will no longer be Markovian. We parameterize economies by node-dependent perturbations $w(\sigma) \in \mathcal{W} \subset \mathbb{R}^N$ and write $\mathcal{E}((w(\sigma))_{\sigma\in\Sigma})$ for a given (nonstationary) perturbed economy. In the original economy $w(\sigma) = 0$ for all

$\sigma \in \Sigma$. The vector $w(\sigma) = (w^e(\sigma), w^u(\sigma), w^f(\sigma))$ may contain perturbations of endowments, preferences, and production functions. For the error analysis of the dynamic models in Sections 5 and 6 we use the following perturbations.

Endowments: For $\sigma \in \Sigma$, $w^e(\sigma)$ denotes additive perturbations of the endowments of those individuals who are active in markets at node σ. The perturbed individual endowment of an agent i is then $\tilde{e}^i(\sigma) = e^i(\sigma) + w^{ei}(\sigma)$.

Preferences: We assume throughout the paper that preferences can be represented by a time-separable expected utility function. We consider linear additive perturbations to Bernoulli utilities (as is often done in general equilibrium analysis; see, e.g., Mas-Colell (1985)). We assume that for an infinitely lived agent i there exists a strictly increasing, strictly concave, and differentiable Bernoulli function $u^i : \mathbb{R}^L_{++} \times \mathcal{S} \to \mathbb{R}$ such that

$$U^i(x) = \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi(s^t) u^i(x(s^t), s_t).$$

Agents have common beliefs Π and discount factors β, and in the original unperturbed economy, Bernoulli utilities only depend on the current shock. Given $u^i(x, s_t)$ and a utility perturbation $w^{ui}(s^t) \in \mathbb{R}^L$, the perturbed Bernoulli utility is

$$\tilde{u}^i(x, s^t) = u^i(x, s^t) + w^{ui}(s^t) \cdot x.$$

These perturbations are difficult to interpret economically since they are not invariant under affine transformations of the original Bernoulli function, but using such linear utility perturbations simplifies the exposition and the notation. We show below how to compute economically meaningful error bounds from these perturbations and properties of the Bernoulli function u^i.

Production Functions: We assume in Section 5 that at each node s^t there is an aggregate technology described by a production function $f_{s_t} : \mathbb{R}^{l_1} \to \mathbb{R}$. We consider linear perturbations and write at node s^t,

$$\tilde{f}_{s_t}(y(s^t)) = f_{s_t}(y(s^t)) + w^f(s^t) \cdot y(s^t).$$

3.1.2. *Close-By Economic Agents*

It is crucial for the analysis that the suggested perturbations of exogenous parameters result in economies that are close-by to the original economy in a meaningful way. The first step in our argument is to define an appropriate topology on the space of possibly perturbed economies. We choose the sup-norm to measure the size of perturbations along the event tree, since we want the perturbed economies to stay as close by as possible to the original economy. Throughout the paper, for a vector $x \in \mathbb{R}^n$, $\|x\|$ denotes the sup-norm, $\|x\| = \max\{|x_1|, \ldots, |x_n|\}$.

Define the space of perturbations to be

$$\ell_\infty(\Sigma, \mathcal{W}) = \Big\{ w = \big(w^e(\sigma), w^u(\sigma), w^f(\sigma)\big) : \sup_{(\sigma,w)\in\Sigma\times\mathcal{W}} \|w(\sigma)\| < \infty \Big\},$$

with $\|x\| = \sup_{\sigma\in\Sigma} \|x(\sigma)\|$ for a sequence $x \in \ell_\infty$. Naturally, a perturbed economy is close to the original economy if the sup-norm of the perturbations is small. For the endowments in the perturbed economy, this obviously means that they are close to the original endowments at all nodes of the tree.

While small differences in individual endowments are easy to understand, differences in utility functions and production functions are more difficult to interpret. In particular, it is obviously sensible to think of utility perturbations in terms of the implied difference in agents' underlying preferences (and not in "utils" as implied by the linear additive perturbations). Following Postlewaite and Schmeidler (1981) and Debreu (1969) we use the Hausdorff distance between sets, d^H, to quantify closeness of two preferences P and P'. The distance between two preferences is

$$d^H(P, P') = \max\Big\{ \sup_{(x,y)\in P} \Big(\inf_{(x',y')\in P'} \|(x, y) - (x', y')\| \Big), \\ \sup_{(x',y')\in P'} \Big(\inf_{(x,y)\in P} \|(x, y) - (x', y')\| \Big) \Big\}.$$

Linearly perturbed utility

$$\tilde{U}^i(x) = \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi(s^t)\big(u^i(x, s^t) + w^{ui}(s^t) \cdot x\big)$$

generally does not represent preferences that are close to the original preferences. However, the following lemma implies that if one finds an exact equilibrium for an economy with utility functions $(\tilde{U}^i)$, there also exists an economy with individual preferences close by to the original preferences for which the same prices and allocation constitute a competitive equilibrium. For simplicity, we assume in the lemma that preferences are homothetic. This allows us to derive explicit bounds on the distance between perturbed preferences and the original preferences.

LEMMA 1: *Suppose that the time-separable expected utility function U represents homothetic preferences P. Given choices $\bar{x} \in \ell_\infty(\Sigma, \mathcal{L})$ and perturbations*

$w \in \ell_\infty$, suppose some $\delta \in \ell_\infty$ satisfies $D_x u(\delta_1(s^t)\bar{x}_1(s^t), \dots, \delta_L(s^t)\bar{x}_L(s^t), s_t) = D_x u(\bar{x}(s^t), s_t) + w(s^t)$ for all s^t. If

$$\bar{x} \in \arg\max \tilde{U}(x) \quad s.t. \quad x \in \mathcal{B},$$

for some convex set $\mathcal{B} \subset \ell_\infty$ with $0 \in \mathcal{B}$, then there exist convex and increasing preferences P' such that whenever $(y, \bar{x}) \in P'$, then $y \notin \mathcal{B}$, i.e., $\bar{x}$ is the best choice in $\mathcal{B}$, and

$$d^H(P, P') \leq \sup_{s^t, l} \bar{x}_l(s^t)\left(1 - \frac{\delta_l(s^t)}{\sup_\sigma \|\delta(\sigma)\|}\right).$$

The lemma, which is proven in the Appendix, shows that for given perturbations to marginal utilities one can construct close-by preferences that support the desired choices. It also shows how to bound the distance between the original preferences and the perturbed preferences. Although the discussion of the perturbations in Section 4 is presented in terms of linear utility perturbations, the reader should keep in mind that such perturbations can be translated to differences in the underlying preferences, which is an economically more meaningful measure.

In applications, it is often tempting to perturb conditional probabilities and node-dependent discount factors. However, such perturbations may lead to preferences that are very far away from the original preferences in our norm. Perturbations in resulting unconditional probabilities may get arbitrarily large for date-events far along the event tree, and therefore marginal rates of substitution for the perturbed preferences will be far from those of the original preferences. The preferences will be far in the Hausdorff distance. For this reason, these perturbations will not be considered in this paper.

3.2. *Equilibrium Equations*

A competitive equilibrium for the economy $\mathcal{E}((w(\sigma))_{\sigma\in\Sigma})$ is a process of endogenous variables $(z(\sigma))_{\sigma\in\Sigma}$ with $z(\sigma) \in \mathcal{Z} \subset \mathbb{R}^M$, which solve agents' optimization problems and clear markets. The set $\mathcal{Z}$ denotes the set of all possible values of the endogenous variables. We refer to the collection of the economy and the endogenous variables, $(\mathcal{E}((w(\sigma))_{\sigma\in\Sigma}), (z(\sigma))_{\sigma\in\Sigma})$, as an *economy in equilibrium*.

In many dynamic economic models an equilibrium can be characterized by a set of equations that relates current-period exogenous and endogenous variables to endogenous variables one period ahead, as well as by a set of equations that restricts current endogenous variables to be consistent with feasibility and optimality. Examples of such conditions are individuals' Euler equations, firms' first-order conditions, and market clearing equations for goods or financial assets. For our error analysis we assume that such a set of equations is given and denote it by

$$h(\bar{s}, \bar{z}, \bar{w}, \kappa, z_1, \ldots, z_S) = 0.$$

The arguments $(\bar{s}, \bar{z}, \bar{w})$ denote the exogenous shock, the endogenous variables, and the perturbations for the current period. For each subsequent exogenous shock s, z_s denotes endogenous variables. The variables $\kappa \in \mathcal{K}$ should be thought of as representing Kuhn–Tucker multipliers or slack variables in inequalities. These variables are used to transform inequalities into equations.

Throughout the analysis, we impose assumptions on preferences and technology, ensuring that $(\mathcal{E}((w(\sigma))_{\sigma\in\Sigma}), (z(\sigma))_{\sigma\in\Sigma})$ is an economy in equilibrium if and only if for all $s^t \in \Sigma$, there exist $\kappa(s^t) \in \mathcal{K}$ such that

$$h\big(s_t, z(s^t), w(s^t), \kappa(s^t), z(s^t 1), \ldots, z(s^t S)\big) = 0.$$

We refer to $h(\cdot) = 0$ as the equilibrium equations.

4. APPROXIMATE EQUILIBRIA AND THEIR INTERPRETATION

The applied computational literature usually refers to recursive equilibria. A recursivity assumption is crucial for computational tractability, because one must find a simple way to represent infinite sequences of allocations and prices. These equilibria are characterized by policy functions that map the current "state" of the economy into choices and prices, and by transition functions that map the state today into a probability distribution over the next period's state. Unfortunately, in dynamic GEI models, recursive equilibria do not always exist, and no nontrivial assumptions are known that guarantee the existence of recursive equilibria (for counterexamples to existence, see, e.g., Hellwig (1982), Kubler and Schmedders (2002), and Kubler and Polemarchakis (2004)). Therefore, one cannot evaluate the quality of a candidate equilibrium by a calculation of how close the computed policy function is to an exact policy function. Instead, we need to define a notion of approximate equilibrium that is general enough to exist in most interesting specifications of the model and is also tractable in the sense that actual approximations in the literature can be interpreted as such equilibria (or at least that these equilibria can be constructed fairly easily from the output of commonly used algorithms). In most popular models, recursive ϵ-equilibria exist (even if exact recursive equilibria fail to exist). We therefore build our error analysis on the concept of recursive ϵ-equilibrium.

The relevant endogenous state space $\Psi \subset \mathbb{R}^D$ depends on the underlying model and is determined by the payoff-relevant, predetermined endogenous variables; that is, by variables sufficient for the optimization of individuals at every date-event, given the prices. The value of the state variables $(s_0, \psi_0) \in \mathcal{S} \times \Psi$ in period 0 is called initial condition and is part of the description of the economy. A recursive ϵ-equilibrium is defined as follows.

DEFINITION 1: A *recursive ϵ-equilibrium* consists of a finite state space Ψ, a policy function $\rho : \mathcal{S} \times \Psi \to \mathbb{R}^{M-D}$, as well as transition functions $\tau_{ss'} : \Psi \to \Psi$, for all $s, s' \in \mathcal{S}$, such that for all states $(\bar{s}, \bar{\psi}) \in \mathcal{S} \times \Psi$, the errors in equilibrium equations at the values implied by policy and transition functions, i.e., at $\bar{z} = (\bar{\psi}, \rho(\bar{s}, \bar{\psi})) \in \mathcal{Z}$ and $(z_1, \dots, z_S)$ with $z_s = (\psi_s, \rho(s, \psi_s))$, $\psi_s = \tau_{\bar{s}s}(\bar{\psi})$ for all $s \in \mathcal{S}$, are below ϵ. That is, they satisfy $\min_{\kappa \in \mathcal{K}} \|h(\bar{s}, \bar{z}, \mathbf{0}, \kappa, z_1, \dots, z_S)\| < \epsilon$.

A recursive ϵ-equilibrium consists of a discretized state space as well as of policy and transition functions, which imply that errors in equilibrium equations are always below ϵ. In most contexts it will be straightforward to derive the transition function from the policy function. For example, in a finance economy, the beginning-of-period portfolio holdings constitute the endogenous state. The policy function assigns new portfolio holdings, which then form the endogenous state next period. In these cases the recursive ϵ-equilibrium is completely characterized by the policy function.

We define the state space as a finite collection of points in order to verify whether a candidate solution constitutes an ϵ-equilibrium. With this definition, the verification involves checking only a finite number of inequalities. Some popular recursive methods (see Judd (1998)) rely on smooth approximations of policy and transition functions using orthogonal polynomials or splines, but we can always extract a recursive ϵ-equilibrium with a discretized state space from such smooth approximations.

Recursive methods enable us to approximate an infinite-dimensional equilibrium by a finite set. Given an initial value of the shock, s_0, and initial values for the endogenous state, ψ_0, a recursive ϵ-equilibrium assigns a value of endogenous variables to any node in the infinite event tree: For any node s^t, the value of the endogenous state is given by $\psi(s^t) = \tau_{s_{t-1}s_t}(\psi(s^{t-1}))$, and the value of the other endogenous variables is given by $\rho(s_t, \psi(s^t))$. We call the resulting stochastic process an ϵ-equilibrium process and write $(z^\epsilon(\sigma))_{\sigma \in \Sigma}$. It will be useful to define ϵ-equilibrium sets, $\mathcal{F} = (\mathcal{F}_1, \dots, \mathcal{F}_S)$ to be the graph of the policy function, so $\mathcal{F}_s \subset \mathcal{Z}$, $\mathcal{F}_s = \text{graph}(\rho(s, \cdot))$.

4.1. *Error Analysis*

Judd (1992) and den Haan and Marcet (1994) suggest evaluating the quality of a candidate solution by using Euler equation residuals. In these methods, relative maximal errors in Euler equations of ϵ usually imply that the solution describes a recursive ϵ-equilibrium. Unfortunately, the example in Section 2 shows that the computed ϵ-equilibrium may be far away from an exact equilibrium for the economy, no matter how small ϵ is. In other words, we cannot perform a pure forward error analysis. As a consequence we perform a mixed forward–backward error analysis and interpret ϵ-equilibria as approximations to exact equilibria of close-by economies. In infinite-horizon models, the question now becomes, what is meant by an approximation to an exact equilibrium?

Ideally a recursive ϵ-equilibrium would generate an ϵ-equilibrium process that is close by to a competitive equilibrium for a close-by economy at all date events. If this were the case, one could find small perturbations of exogenous parameters such as endowments and preferences of the original economy so that the perturbed economy has a competitive equilibrium that is well approximated by the ϵ-equilibrium process at *each* node of the event tree. This idea is formalized in the following definition of approximate equilibrium.

DEFINITION 2: Given an economy $\mathcal{E}$, an ϵ-equilibrium process $(z^\epsilon(\sigma))_{\sigma\in\Sigma}$ is called *path-approximating* with error δ if there exists an economy in equilibrium, $(\mathcal{E}((w(\sigma))_{\sigma\in\Sigma}), (\hat{z}(\sigma))_{\sigma\in\Sigma})$ with $\sup_{\sigma\in\Sigma}\|w(\sigma)\| < \delta$ and $\sup_{\sigma\in\Sigma}\|z^\epsilon(\sigma) - \hat{z}(\sigma)\| < \delta$.

In models with finitely lived agents, e.g., OLG models, ϵ-equilibria will usually be path-approximating. However, in models where agents are infinitely lived, we cannot expect that a recursive ϵ-equilibrium gives rise to a process that path-approximates a close-by economy in equilibrium. If agents make small errors in their choices each period, these errors are likely to propagate over time, and after sufficiently many periods the ϵ-equilibrium allocation will be far away from the exact equilibrium allocation. The following simple example illustrates the ease of constructing ϵ-equilibria that do not path-approximate an economy in equilibrium, no matter how small ϵ is.

EXAMPLE 2: Consider an infinite-horizon exchange economy with two infinitely lived agents, a single commodity, and no uncertainty. Suppose that agents have identical initial endowments $e^1 = e^2 > 0$ in each period and identical preferences with $u_i(c_t) = \log(c_t)$ and with a common discount factor $\beta \in (0, 1)$. There is a consol in unit net supply that pays 1 unit of the consumption good each period. The price of the consol is q_t, portfolios are θ^i_t. Each agent i, $i = 1, 2$, faces a short-sale constraint $\theta^i_t \geq 0$ for all t.

Let the endogenous variables be $z = ((\theta^i_-, \theta^i, c^i, m^i)_{i=1,2}, q)$. Admissible perturbations are $w = (w^{ui}, w^{ei})_{i=1,2} \in \mathbb{R}^4$, so we allow for perturbations both in endowments and in preferences. The equilibrium equations $h(\bar{z}, \bar{w}, \kappa, z) = 0$ with $h = (h^1, \dots, h^6)$ are

$$
\begin{aligned}
h^1 &= -1 + \beta\frac{(1+q)m^i}{\bar{q}\bar{m}^i} + \kappa^i && (i = 1, 2),\\
h^2 &= \kappa^i\bar{\theta}^i && (i = 1, 2),\\
h^3 &= \bar{c}^i - \bar{\theta}^i_-(\bar{q} + 1) + \bar{\theta}^i\bar{q} - (e^i + \bar{w}^{ei}) && (i = 1, 2),\\
h^4 &= \theta^i_- - \bar{\theta}^i && (i = 1, 2),\\
h^5 &= \bar{m}^i - (u'_i(\bar{c}^i) + \bar{w}^{ui}) && (i = 1, 2),\\
h^6 &= \bar{\theta}^1 + \bar{\theta}^2 - 1.
\end{aligned}
$$

The natural endogenous state space for this economy consists of beginning-of-period consol holdings. We build market clearing into the state space and only consider θ^1_-, θ^2_- with $\theta^1_- + \theta^2_- = 1$. We write $\psi = \theta^1_-$ to represent a typical state of the economy, implicitly assuming market clearing. For any initial condition $\psi_0 \in (0, 1)$ the unique exact equilibrium is no trade in the consol, with each agent consuming $\theta^i_{0-} + e^i$ every period, where $\theta^1_{0-} = \psi_0 = 1 - \theta^2_{0-}$. The consol price is $q_t = \frac{\beta}{1-\beta}$ for all $t \geq 0$.

Now suppose each period agent 1 sells a small amount of the consol to agent 2. As a result, agent 1's consumption converges to e^1 while the consumption of agent 2 converges to $e^2 + 1$. There is no economy with close-by endowments for which this allocation is an approximate equilibrium allocation. We can construct a recursive ϵ-equilibrium as follows. Define

$$\tau(\psi) = \begin{cases} \psi - \delta, & \text{if } \psi > \delta, \\ \psi, & \text{otherwise.} \end{cases}$$

Define $q = \rho_q(\psi) = \frac{\beta}{1-\beta}$, $\theta^1 = \rho_{\theta^1}(\psi) = \tau(\psi)$, and $c^1(\psi) = e^1 + \psi(q+1) - \theta q$, $c^2(\theta) = 1 + e^1 + e^2 - c^1(\theta)$. These functions describe a recursive ϵ-equilibrium as long as $0 \leq \delta < \epsilon(1-\beta)e^1$. Except for the Euler equations h^1, all equilibrium equations hold with equality. For $\theta_- = \psi > 2\delta$, the error in Euler equations for agent 1 is given by

$$\|h^1\| = \left| -1 + \frac{e^1 + (\theta + \delta)(q+1) - \theta q}{e^1 + \theta(q+1) - \theta q + \delta q} \right| < \frac{\delta(q+1)}{e^1}.$$

For $2\delta \geq \psi > \delta$, we have

$$\|h^1\| = \left| -1 + \frac{e^1 + (\theta + \delta)(q+1) - \theta q}{e^1 + \theta(q+1) - \theta q} \right| \leq \frac{\delta(q+1)}{e^1}$$

and finally for $\psi < \delta$, we have $h^1 = 0$. The argument for agent 2 is analogous. For the initial condition $\psi_0 = 0.5$, the constructed recursive equilibrium yields an ϵ-equilibrium process which, in the sup-norm, is far from any exact equilibrium of a close-by economy. Figure 3 shows the exact equilibrium and the ϵ-equilibrium process.

Note that this phenomenon is a general problem that does not only occur in economies with incomplete markets. The same phenomenon can even arise for an approximate solution to a single-agent decision problem. In the applied literature this problem is commonly addressed by using a weaker notion of approximate equilibrium:[4] A computed solution is considered a good approximation if the computed policy function is close by the true policy function.

[4]A notable exception is Santos and Peralta-Alva (2002) who derive sufficient conditions for sample-path stability in a representative agent model.

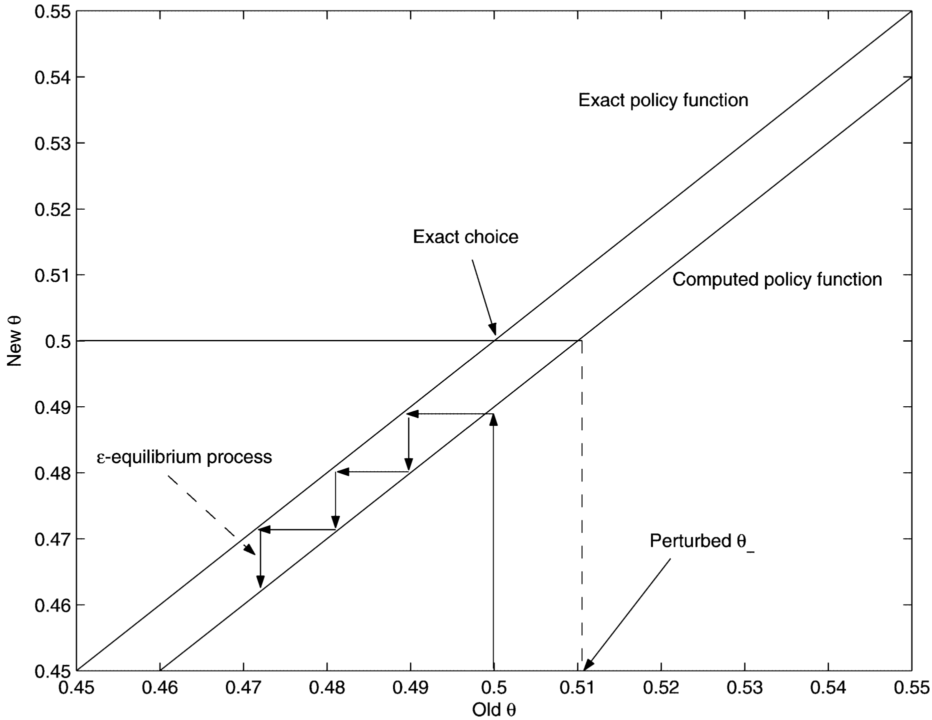

FIGURE 3.—Weak approximation.

We generalize this idea and apply it to our general framework. Instead of requiring that the exact equilibrium process is well approximated by the ϵ-equilibrium process, we merely require that it is well approximated by the ϵ-equilibrium set: For each node $s^t \in \Sigma$, given the value of the endogenous state of the exact equilibrium process, there is a state close by such that the value of the (ϵ-equilibrium) policy function at this state is close to the value of endogenous variables of the exact equilibrium. The following definition formalizes this weaker notion of approximation.

DEFINITION 3: A recursive ϵ-equilibrium with equilibrium set $\mathcal{F}$ for the economy $\mathcal{E}$ is called *weakly approximating* with error δ if there exists an economy in equilibrium $(\mathcal{E}((w(\sigma))_{\sigma\in\Sigma}), (\hat{z}(\sigma))_{\sigma\in\Sigma})$, with $\sup_{\sigma\in\Sigma} \|w(\sigma)\| < \delta$ such that for all $s^t \in \Sigma$ and all $\hat{z}(s^t)$ there exists a $z \in \mathcal{F}_{s_t}$ which satisfies $\|z - \hat{z}(s^t)\| < \delta$.

Intuitively, the definition requires that for the recursive ϵ-equilibrium the policy function is close to the policy function of an exact recursive equilibrium (of a close-by economy). In the models we consider in this paper existence of exact recursive equilibria cannot be established. Therefore, we state the definition in terms of competitive equilibria $\hat{z}(\sigma)_{\sigma\in\Sigma}$ of the close-by economy $\mathcal{E}((w(\sigma))_{\sigma\in\Sigma})$.

This condition is much weaker than requiring that the ϵ-equilibrium process path-approximates an economy in equilibrium. This is to be expected, since closeness in policy functions generally does not yield any implications about how close equilibrium allocations are, even in models where recursive equilibria do exist. The definition only requires that there exists some process with values in $\mathcal{F}$ that approximates the exact equilibrium but does not explicitly state how to construct this process.

The ϵ-equilibrium in Example 2 weakly approximates the exact equilibrium. For any initial condition the exact equilibrium (for an economy that does not need to be perturbed) involves no trade and an asset price of $\beta/(1-\beta)$. For the same initial condition the recursive ϵ-equilibrium implies the same asset price and trading of less than δ units of the asset.

In general, of course, verifying that an ϵ-equilibrium satisfies Definition 3 will not be as straightforward as in this example, because the exact equilibrium is not known. We explain this problem more carefully with a concrete example in Section 6. The strategy there is illustrated in Figure 3. Given an approximate policy function and an initial state (in the figure, $\theta_{-} = 0.5$), we try to find a point in the state space where the value of the approximate policy function equals the value of the exact policy function at the initial state.

5. A MODEL WITH OVERLAPPING GENERATIONS AND PRODUCTION

As a first application of our methods we consider a model of a production economy with overlapping generations. In this model the ϵ-equilibrium process path-approximates an economy in equilibrium and we derive bounds on the distance between the close-by economy and the specified economy.

5.1. *The Economy*

Recall that each node of the event tree represents a history of exogenous shocks to the economy, $\sigma = s^t = (s_0, s_1, \ldots, s_t)$. The shocks follow a Markov chain with finite support $\mathcal{S}$ and with transition matrix Π. Three commodities are traded at each date event: labor, a consumption good, and a capital good that can only be used as input to production and yields no utility. The economy is populated by overlapping generations of agents that live for $N+1$ periods, $a = 0, \ldots, N$. An agent is fully characterized by the node in which she is born (s^t). When there is no ambiguity we index the agent by the date of birth. An agent born at node s^t has nonnegative labor endowment over her life cycle, which depends on the exogenous shock and age, $\mathbf{e}^a(s)$, for ages $a = 0, \ldots, N$ and shocks $s \in \mathcal{S}$. Agents have no endowments in the capital and the consumption good. The price of the consumption good at each date event is normalized to 1, the price of capital is denoted by $p_k(s^t)$, and the market wage is $p_l(s^t)$. The agent has an intertemporal, von Neumann–Morgenstern utility function over consumption, c, and leisure, l, over his life cycle,

$$U^{s^t} = \mathrm{E}_{s^t} \sum_{a=0}^{N} \beta^a u\big(c(s^{t+a}), l(s^{t+a}); s_{t+a}\big).$$

The Bernoulli utility u may depend on the current shock s and is strictly increasing, strictly concave, and continuously differentiable. We denote the partial derivatives by u_c and u_l.

Households have access to a storage technology. They can use one unit of the consumption good to obtain one unit of the capital good in the next period. We denote the investment of household t at node s^{t+a} into this technology by $\theta^t(s^{t+a})$. All agents are born with zero assets, $\theta^t(s^{t-1}) = 0$. We do not restrict investments to be nonnegative, thus allowing households to borrow against future labor income.

There is a single representative firm, which in each period uses labor and capital to produce the consumption good according to a constant returns to scale production function $f(K, L; s)$, given shock $s \in \mathcal{S}$. Firms make decisions on how much capital to buy and how much labor to hire after the realization of the shock s_t, and so they face no uncertainty and simply maximize current profits. At time t the household sells all its capital goods accumulated from last period, $\theta^\sigma(s^{t-1})$, to the firm for a market price $p_k(s^t) > 0$.

For given initial conditions s_0, $((\theta^t(s_{-1}))_{t=-N}^{0}$ a competitive equilibrium is a collection of choices for households $(c^t(s^{t+a}), l^t(s^{t+a}), \theta^t(s^{t+a}))_{a=0}^{N}$, for the representative firm $(K(s^t), L(s^t))$, as well as prices $(p_k(s^t), p_l(s^t))$ such that households and the firm maximize and markets clear for all s^t.

We want to characterize competitive equilibria by equilibrium equations. We define the endogenous variables at some node to consist of individuals' investments from the previous period, $\theta_- = (\theta_-^0, \dots, \theta_-^N)$, new investments, $\theta = (\theta^0, \dots, \theta^N)$, consumption and leisure choices, $c = (c^0, \dots, c^N)$ and $l = (l^0, \dots, l^N)$ as well as Lagrange multipliers $\lambda = (\lambda^0, \dots, \lambda^N)$, the firm's choices K, L, and spot prices, (p_l, p_k), so $z = (\theta_-, \theta, c, l, \lambda, K, L, p_k, p_l)$. We build bounds and normalizations into the admissible endogenous variables, i.e., we only consider z for which $\theta_-^0 = 0$, $\theta^N = 0$, $c, l, \lambda \geq 0$, $K, L \geq 0$. We consider perturbations in individual endowments, in preferences and in production functions, i.e., define $w(\sigma) = (w^e(\sigma), w^u(\sigma), w^f(\sigma)) \in \mathbb{R}^{N+1} \times \mathbb{R}^{2(N+1)} \times \mathbb{R}^2$ to be perturbations in endowments and preferences across all agents alive and in the production function at node σ. We write the perturbed Bernoulli function of an agent of age a as

$$u(c, l, s) + w^{ua} \cdot \begin{pmatrix} c \\ l \end{pmatrix} \quad \text{for} \quad w^{ua} \in \mathbb{R}^2.$$

Production functions are perturbed in a similar fashion:

$$f(K, L, s) + w^f \cdot \begin{pmatrix} K \\ L \end{pmatrix}.$$

Equilibrium is characterized by equilibrium equations; $\bar{s}$, $\bar{z}$, $\bar{w}$, and $z(1), \ldots, z(S)$ are consistent with equilibrium if and only if $h(\bar{s}, \bar{z}, \bar{w}, z(1), \ldots, z(S)) = 0$ with

$$h = \begin{cases} \theta_-^a(s) - \bar{\theta}^{a-1} & (a = 1, \ldots, N, s \in \mathcal{S}) \quad (h^1), \\ -\bar{\lambda}^{a-1} + \beta \sum_{s \in \mathcal{S}} \Pi(s|\bar{s}) \lambda^a(s) p_k(s) & (a = 1, \ldots, N) \quad (h^2), \\ \bar{\theta}_-^a \bar{p}_k + (\mathbf{e}^a(\bar{s}) + \bar{w}^{ea} - \bar{l}^a) \bar{p}_l - \bar{\theta}^a - \bar{c}^a & (a = 0, \ldots, N) \quad (h^3), \\ u_c(\bar{c}^a, \bar{l}^a, \bar{s}) + \bar{w}_1^{ua} - \bar{\lambda}^a & (a = 0, \ldots, N) \quad (h^4), \\ u_l(\bar{c}^a, \bar{l}^a, \bar{s}) + \bar{w}_2^{ua} - \bar{\lambda}^a \bar{p}_l & (a = 0, \ldots, N) \quad (h^5), \\ \bar{p}_k - (f_K(\bar{K}, \bar{L}, \bar{s}) + \bar{w}_1^f) & (h^6), \\ \bar{p}_l - (f_L(\bar{K}, \bar{L}, \bar{s}) + \bar{w}_2^f) & (h^7), \\ \sum_{a=0}^{N} \bar{\theta}^a - \bar{K} & (h^8), \\ \sum_{a=0}^{N} (\mathbf{e}^a(\bar{s}) + \bar{w}^{ea} - \bar{l}^a) - \bar{L} & (h^9). \end{cases}$$

We denote equation $h^i(\cdot) = 0$ by $(h_{\text{olg}} i)$ for $i = 1, \ldots, 9$. Under standard assumptions on preferences and the production function, which guarantee that first-order conditions are necessary and sufficient, a competitive equilibrium can be characterized by these equations. The natural endogenous state space for a recursive equilibrium consists of individuals' beginning-of-period holdings in the capital good θ_-. Kubler and Polemarchakis (2004) prove the existence of recursive ϵ-equilibria. The equilibrium values of all variables are given by a policy function ρ. For example, we write $\rho_K(s, \theta_-)$ for the policy term that determines aggregate capital K. For a given recursive ϵ-equilibrium, the transition function is given by equation $(h_{\text{olg}} 1)$ which we assume to hold exactly, i.e., $\theta_-^a(s) = \bar{\theta}^{a-1}$ for all a, s. These functions then also determine an ϵ-equilibrium process.

5.2. *Error Analysis*

Given an ϵ-equilibrium $\mathcal{F}$, the objective of our error analysis is to provide uniform bounds on necessary perturbations of the underlying economy and on the distance of the ϵ-equilibrium process $(z^\epsilon(\sigma))$ from the exact equilibrium of the perturbed economy. We first perturb only preferences, i.e., we construct a close-by economy with $w^e = w^f = 0$, but allow w^u to be nonzero, and show that the ϵ-equilibrium process is close to the exact equilibrium of this economy. Subsequently we outline one different set of perturbations. We show that it suffices to perturb technology and endowments, holding preferences fixed.

We show how to construct bounds on forward errors, Δ^F, and bounds on backward errors, Δ^B, such that there exists a close-by economy with an exact equilibrium $(\hat{z}(\sigma))_{\sigma\in\Sigma}$ that satisfies $\|\hat{z}(\sigma) - z^\epsilon(\sigma)\| < \Delta^F$ and $\|w(\sigma)\| < \Delta^B$ for all $\sigma \in \Sigma$, including

$$\hat{\theta}_-(\sigma) = \theta^\epsilon_-(\sigma), \quad \hat{\theta}(\sigma) = \theta^\epsilon(\sigma), \quad \hat{l}(\sigma) = l^\epsilon(\sigma),$$
$$\hat{K}(\sigma) = K^\epsilon(\sigma), \quad \hat{L}(\sigma) = L^\epsilon(\sigma).$$

By allowing for backward errors, only prices, consumption values, and Lagrange multipliers need to be perturbed. All other endogenous variables in the exact equilibrium $(\hat{z}(\sigma))_{\sigma\in\Sigma}$ take the corresponding values from the ϵ-equilibrium process. Intuitively, our perturbation procedure is best described as a "backward substitution" approach. For every state $(\bar{s}, \bar{z})$ with $\bar{z} \in \mathcal{F}_{\bar{s}}$ we reduce the errors equation by equation to zero by perturbing some endogenous variables. Of course, we have to keep track of how perturbations in one equation affect the errors in other equations where perturbed variables also appear. The possibility of using backward errors is crucial to ensure that this backward substitution approach successfully sets all equilibrium equations simultaneously to zero. We now provide the technical details.

Consider a particular $\bar{s}$, $\bar{z} \in \mathcal{F}_{\bar{s}}$ and values of next period variables $(z(1), \dots, z(S)) \in \mathcal{F}$, where

$$z(s) = \big(\bar{\theta}, \rho_\theta(s, \bar{\theta}), \rho_c(s, \bar{\theta}), \rho_l(s, \bar{\theta}), \rho_\lambda(s, \bar{\theta}),$$
$$\rho_K(s, \bar{\theta}), \rho_L(s, \bar{\theta}), \rho_{p_k}(s, \bar{\theta}), \rho_{p_l}(s, \bar{\theta})\big).$$

By definition, these points satisfy $\|h(\bar{s}, \bar{z}, 0, z(1), \dots, z(S))\| < \epsilon$. We assume, without loss of generality, that for all $\bar{s} \in \mathcal{S}$ and all $\bar{z} \in \mathcal{F}_{\bar{s}}$, $\sum_{a=0}^N (\mathbf{e}^a(\bar{s}) - \bar{l}^a) = \bar{L}$ and $\sum_{a=0}^N \bar{\theta}^a - \bar{K} = 0$. Both conditions can be built into the construction of $\bar{z}$. Consequently, the equations given by $(h_{\text{olg}}8)$ and $(h_{\text{olg}}9)$ hold exactly.

To set other equations exactly equal to zero, we must perturb values in $\bar{z}$, which in turn affect other equations. We keep track of the possibly increased errors in the other equations and adjust them accordingly. Moreover, we perturb $\lambda(s)$, $s = 1, \dots, S$, for equality in $(h_{\text{olg}}2)$. Such perturbations then fix $\lambda(s)$ and therefore enter the Euler equations $(h_{\text{olg}}2)$ and equations $(h_{\text{olg}}4)$ and $(h_{\text{olg}}5)$ for the subsequent period. As a result, we must be careful how we perform our perturbation analysis to set those equations equal to zero.

We adjust prices in order to set equations ($h_{\text{olg}}6$) and ($h_{\text{olg}}7$) to zero. Maximal errors in prices are simply given by

$$\Delta_1 = \max_{\bar{s}\in\mathcal{S},\bar{z}\in\mathcal{F}_{\bar{s}}} \|\bar{p}_k - f_K(\bar{K},\bar{L},\bar{s})\| \quad \text{and}$$

$$\Delta_2 = \max_{\bar{s}\in\mathcal{S},\bar{z}\in\mathcal{F}_{\bar{s}}} \|\bar{p}_l - f_L(\bar{K},\bar{L},\bar{s})\|.$$

For equations ($h_{\text{olg}}3$) to hold exactly, individual consumption values $\bar{c}^a$ must be perturbed. Given the corrected prices, maximal necessary perturbations in individuals' consumptions are then

$$\Delta_3 = \max_{\bar{s}\in\mathcal{S},\bar{z}\in\mathcal{F}_{\bar{s}},a=0,\dots,N} \big\|\bar{\theta}^a_- f_K(\bar{K},\bar{L},\bar{s}) + (\mathbf{e}^a(\bar{s}) - \bar{l}^a) f_L(\bar{K},\bar{L},\bar{s}) - \bar{\theta}^a - \bar{c}^a\big\|.$$

In practice the perturbations in prices and consumptions will be tiny since the respective equations can be solved very precisely. More significant errors may arise, however, through cumulative perturbations in λ over an agent's life-span. Equations ($h_{\text{olg}}2$) show that if the current λ has been perturbed away from $\tilde{\lambda}$ for last period's Euler equation, the necessary perturbations in $\lambda(1),\dots,\lambda(S)$ might propagate over time. To analyze this problem, define

$$\delta_2^a = \max_{\bar{s},\bar{z}\in\mathcal{F}_{\bar{s}}} \left\| -\bar{\lambda}^{a-1} + \beta\sum_{s\in\mathcal{S}} \Pi(s,\bar{s})\rho_{\lambda^a}(s,\bar{\theta}) f_K\big(\rho_K(s,\bar{\theta}),\rho_L(s,\bar{\theta}),s\big)\right\|,$$

where the prices have already been replaced by their perturbed values. Note that δ_2^a is an upper bound on errors in equations ($h_{\text{olg}}2$) only if $\tilde{\lambda} = \bar{\lambda}$. For the general case where $|\tilde{\lambda} - \bar{\lambda}| < \delta_1$, the maximal error is bounded by $\delta_1 + \delta_2^a$, by the triangle inequality.

The perturbations of Lagrange multipliers are mirrored by perturbations of marginal utilities in order to ensure that equations ($h_{\text{olg}}4$) and ($h_{\text{olg}}5$) hold exactly. Define

$$\Delta_4 = \max_{\bar{s},\bar{z}\in\mathcal{F}_{\bar{s}},a=1,\dots,N} \big\|u_c\big(\bar{\theta}^a_- f_K(\bar{K},\bar{L},\bar{s}) + (\mathbf{e}^a(\bar{s}) - \bar{l}^a) f_L(\bar{K},\bar{L},\bar{s}) - \bar{\theta}^a, \bar{l}^a,\bar{s}\big) - \bar{\lambda}^a\big\|,$$

$$\Delta_5 = \max_{\bar{s},\bar{z}\in\mathcal{F}_{\bar{s}},a=1,\dots,N} \big\|u_l\big(\bar{\theta}^a_- f_K(\bar{K},\bar{L},\bar{s}) + (\mathbf{e}^a(\bar{s}) - \bar{l}^a) f_L(\bar{K},\bar{L},\bar{s}) - \bar{\theta}^a, \bar{l}^a,\bar{s}\big) - \bar{\lambda}^a f_L(\bar{K},\bar{L},\bar{s})\big\|.$$

Given that necessary perturbations in λ are bounded by $\delta_1 + \delta_2^a$, maximal necessary perturbations in Bernoulli utilities are then bounded by $\delta_1 + \delta_2^a + \Delta_4$ for $\bar{w}_1^u$ and $\Delta_5 + (\delta_1 + \delta_2^a)P_l^{\max}$ for $\bar{w}_2^u$, where $P_l^{\max} = \max_{\bar{s}\in\mathcal{S},\bar{z}\in\mathcal{F}_{\bar{s}}} f_L(\bar{K},\bar{L},\bar{s})$.

Can we find a δ_1 such that for all $\sigma \in \Sigma$ and the exact equilibrium values $\hat{\lambda}$, $\|\hat{\lambda}(\sigma) - \bar{\lambda}\| < \delta_1$? To bound the perturbations of Lagrange multipliers over time, the crucial insight is that we can set $\hat{\lambda}^0 = \bar{\lambda}^0$ and then only must keep track of the propagation of perturbations over an agents' lifetime. For this, define

$$M = \min_{\bar{s}, \bar{z} \in \mathcal{F}_{\bar{s}}} \beta \sum_{s \in \mathcal{S}} \Pi(s, \bar{s}) \rho_{p_k}(\bar{\theta}, s) \quad \text{and} \quad \Delta_6 = \delta_2 \sum_{a=1}^{N} \frac{\delta_2^a}{M^{N-a+1}}.$$

Cumulative perturbations in Lagrange multipliers over an agent's life-span are then bounded by Δ_6. As a result, the cumulative perturbations in Bernoulli utilities are bounded by $\Delta_6 + \Delta_4$ for $\bar{w}_1^u$ and $\Delta_5 + \Delta_6 P_l^{\max}$ for $\bar{w}_2^u$. We have now established the maximal necessary perturbations of the ϵ-equilibrium process in order to obtain an exact equilibrium for a close-by economy.

ERROR BOUND 1: Consider an ϵ-equilibrium process $(z^\epsilon(\sigma))$ for the OLG production economy with $z^\epsilon(\sigma) = (\theta^\epsilon_-(\sigma), \theta^\epsilon(\sigma), c^\epsilon(\sigma), l^\epsilon(\sigma), \lambda^\epsilon(\sigma), K^\epsilon(\sigma), L^\epsilon(\sigma), p_k^\epsilon(\sigma), p_l^\epsilon(\sigma))$. There exists an economy in equilibrium $(\mathcal{E}((w(\sigma))_{\sigma\in\Sigma}), (\hat{z}(\sigma))_{\sigma\in\Sigma})$ with backward perturbations

$$\left\|(w_1^u(\sigma))_{\sigma\in\Sigma}\right\| \leq \Delta_6 + \Delta_4 \quad \text{and} \quad \left\|(w_2^u(\sigma))_{\sigma\in\Sigma}\right\| \leq \Delta_5 + \Delta_6 P_l^{\max}$$

and forward perturbations

$$\left\|(p^\epsilon(\sigma) - \hat{p}(\sigma))_{\sigma\in\Sigma}\right\| \leq \max(\Delta_1, \Delta_2), \quad \left\|(c^\epsilon(\sigma) - \hat{c}(\sigma))_{\sigma\in\Sigma}\right\| \leq \Delta_3,$$
$$\|(\lambda^\epsilon(\sigma) - \hat{\lambda}(\sigma))_{\sigma\in\Sigma}\| \leq \Delta_6.$$

The remaining equilibrium variables have the (unperturbed) values from the ϵ-equilibrium process, so $\hat{\theta}(\sigma) = \theta^\epsilon(\sigma)$, $\hat{l}(\sigma) = l^\epsilon(\sigma)$, $\hat{K}(\sigma) = K^\epsilon(\sigma)$, and $\hat{L}(\sigma) = L^\epsilon(\sigma)$.

In the statement we report all errors as absolute as opposed to relative errors. In many economic applications it is more meaningful to report relative errors, but the same analysis applies except that all errors have to be taken to be relative errors. Furthermore, Lemma 1 transforms the bounds on utility perturbations into corresponding perturbations of preferences.

If the economic application commands that preferences have to be held fixed, that is, $w_1^{ua}(s) = w_2^{ua}(s) = 0$ for all a, s, then we must perturb both individual endowments and production functions to achieve an approximation to an exact equilibrium. We start with the ϵ-equilibrium process $(z^\epsilon(\sigma))$ and ask under which conditions this process approximates an equilibrium process for an economy where endowments and technology are perturbed but utility functions are not. We use the same strategy as above to bound necessary perturbations of λ over time and obtain a value for Δ_6. However, now, for $(h_{\text{olg}}4)$

and ($h_{\mathrm{olg}}5$) to hold with equality, since $w^u = 0$, consumption and leisure choices have to be different. Bounds δ^c and δ^l on the perturbations in (c, l) can be obtained through a direct calculation:

$$u_c(\bar{c}^a + \delta^c, \bar{l}^a + \delta^l, \bar{s}) = u_c(\bar{c}^a, \bar{l}^a, \bar{s}) + \Delta_6,$$
$$u_l(\bar{c}^a + \delta^c, \bar{l}^a + \delta^l, \bar{s}) = u_l(\bar{c}^a, \bar{l}^a, \bar{s}) + \Delta_6 P_l^{\max}.$$

This directly implies the necessary w^{ea} to make ($h_{\mathrm{olg}}3$) hold with equality. With different l^a and $w^{ea} \neq 0$, ($h_{\mathrm{olg}}9$) will no longer hold with equality, and one must perturb $\bar{L}$. Finally, to ensure that ($h_{\mathrm{olg}}6$) and ($h_{\mathrm{olg}}7$) hold with equality, given the original prices (which do not need to be perturbed), one must perturb production functions via w^f. Note that we do not perturb endowments of the consumption and capital good, since they are zero.

5.3. *Parametric Examples*

We illustrate the result of Error Bound 1 with an example. There are $S = 2$ shocks, which are i.i.d. with $\pi_s = 0.5$ for $s = 1, 2$. Suppose the risky spot production function is Cobb–Douglas, $f(K, L, s) = \eta(s)K^{\alpha}L^{1-\alpha} + (1-\delta)K$, with $\alpha = 0.36$ and $\delta = 0.7$ for shocks $\eta = (\eta(1), \eta(2)) = (0.85, 1.15)$. Agents live for six periods, $a = 0, 1, \ldots, 5$, and only derive utility from the consumption good. An agent born at shock s^t has utility function

$$U^{s^t} = \mathrm{E}_{s^t} \sum_{a=1}^{N} \beta^{a-1} \frac{(c(s^{t+a-1}))^{1-\gamma}}{1-\gamma}$$

with a coefficient of relative risk aversion $\gamma = 1.5$ and discount factor $\beta = 0.8$. Individual labor endowments are deterministic, $(e^0, e^1, \ldots, e^5) = (1.2, 1.3, 1.4, 1.4, e^4, e^5)$. We consider two different specifications of the endowments for $a = 4, 5$, namely $(e^4, e^5) = (1, 1)$ and $(e^4, e^5) = (0.1, 0.1)$.

Table I reports the number of elements in the ϵ-equilibrium set $\mathcal{F}$ and the maximal errors in equilibrium equations, ϵ, as well as the quantities M, Δ_6, and $P^l_{\max}$ that play key roles in the calculations of the bounds on backward and forward errors. We also report $\Delta c(\epsilon)$ which denotes the maximal consumption-equivalent error in intertemporal Euler equations (see Judd (1998)) and the

TABLE I

ERRORS

(e^4, e^5)	$\#\mathcal{F}$	ϵ	M	Δ_6	$P^l_{\max}$	$\Delta c(\epsilon)$	$d^H(P, P')$
$(1, 1)$	4,193,610	9.97 (−4)	1.065	2.94 (−3)	0.407	1.25 (−4)	2.86 (−3)
$(0.1, 0.1)$	1,855,596	3.75 (−3)	0.828	1.66 (−2)	0.493	2.55 (−4)	3.36 (−3)

bound on $d^H(P, P')$ from Lemma 1, the distance between original preferences and perturbed preferences.

Note that the necessary perturbations in fundamentals are quite small throughout and are not much larger than the equation error ϵ. Although for the second specification, perturbations in utilities are quite large (greater than 10^{-2}), translated to consumption-equivalent perturbations they become small.

6. THE LUCAS MODEL WITH SEVERAL AGENTS

As a second application we consider the model of Duffie et al. (1994, Section 3). This model is a version of the Lucas (1978) asset pricing model with finitely many heterogeneous agents. There are I infinitely lived agents, $i \in \mathcal{I}$, and a single commodity in a pure exchange economy. Each agent $i \in \mathcal{I}$ has endowments $e^i(\sigma) > 0$ at all nodes $\sigma \in \Sigma$, which are time-invariant functions of the shock alone, i.e., there exist functions $\mathbf{e}^i : \mathcal{S} \to \mathbb{R}_+$ such that $e^i(s^t) = \mathbf{e}^i(s_t)$. Agent i has von Neumann–Morgenstern utility over infinite consumption streams $U^i(c) = E_0 \sum_{t=0}^{\infty} \beta^t u_i(c_t)$ for a differentiable, strictly increasing, and concave Bernoulli function u_i which satisfies an Inada condition.

There are J infinitely lived assets in unit net supply. Each asset $j \in \mathcal{J}$ pays shock dependent dividends $d_j(s)$. We denote its price at node s^t by $q_j(s^t)$. Agents trade these assets but are restricted to hold nonnegative amounts of each asset. We denote portfolios by $\theta^i \geq 0$. At the root node s_0, agents hold initial shares $\theta^i(s_{-1})$, which sum to 1.

A competitive equilibrium is a collection $((\hat{c}^i(\sigma), \hat{\theta}^i(\sigma))_{i\in\mathcal{I}}, \hat{q}(\sigma))_{\sigma\in\Sigma}$ such that markets clear and such that agents optimize, i.e.,

$$(\hat{c}^i, \hat{\theta}^i) \in \arg \max_{(c^i, \theta^i) \geq 0} U^i(c^i) \quad \text{s.t.} \quad \forall s^t \in \Sigma$$

$$c^i(s^t) = \mathbf{e}^i(s_t) + \theta^i(s^{t-1})(\hat{q}(s^t) + d(s_t)) - \theta^i(s^t)\hat{q}(s^t).$$

6.1. *The Equilibrium Equations*

We define the current endogenous variables to consist of beginning-of-period portfolio holdings, $\theta_- = (\theta_-^1, \dots, \theta_-^I)$, new portfolio choices, θ, asset prices, q, individuals' consumptions, $c = (c^1, \dots, c^I)$, and individuals' marginal utilities, $m^i = u_i'(c^i)$, i.e., $z = (\theta_-, \theta, q, c, m)$. We again build normalizations into the state space so that $\theta^i, c^i \geq 0$ for all $i \in \mathcal{I}$ and that $\sum_{i\in\mathcal{I}} \theta_-^i = 1$.

For the error analysis, we perturb the per-period utility functions, u^i, as well as individual endowments. (The error analysis is simplified by considering both perturbations, but we show below that, in general, perturbations only in endowments suffice.) We take perturbations to be $2I$ vectors, $w = (w^u, w^e) = (w^{u1}, \dots, w^{uI}, w^{e1}, \dots, w^{eI}) \in \mathbb{R}^I \times \mathbb{R}^I$. The equilibrium equations are then

$h(\bar{s}, \bar{z}, \bar{w}, \kappa, z(1), \ldots, z(S)) = 0$ with

$$h = \begin{cases} -\bar{q}\bar{m}^i + \beta \sum_{s \in \mathcal{S}} \Pi(s|\bar{s})(q(s) + d(s))m^i(s) + \kappa^i & (i \in \mathcal{I}) \quad (h^1), \\ \kappa^i_j \bar{\theta}^i_j & (i \in \mathcal{I}, j \in \mathcal{J}) \quad (h^2), \\ \bar{m}^i - (u'_i(\bar{c}^i) + \bar{w}^{ui}) & (i \in \mathcal{I}) \quad (h^3), \\ \bar{c}^i - \bar{\theta}^i_-(\bar{q} + d(\bar{s})) + \bar{\theta}^i \cdot \bar{q} - (\mathbf{e}^i(\bar{s}) + \bar{w}^{ei}) & (i \in \mathcal{I}) \quad (h^4), \\ \theta^i_-(s) - \bar{\theta}^i & (i \in \mathcal{I}) \quad (h^5), \\ \sum_{i \in \mathcal{I}} \bar{\theta}^i_j - 1 & (i \in \mathcal{I}, j \in \mathcal{J}) \quad (h^6). \end{cases}$$

We denote equation $h^i(\cdot) = 0$ by $(h_{\inf} i)$ for all $i = 1, \ldots, 6$. Duffie et al. (1994) provide conditions on u^i which ensure that these equations are necessary and sufficient for an equilibrium. The natural endogenous state for this economy consists of beginning-of-period portfolios, see, e.g., Heaton and Lucas (1996), although a recursive equilibrium for this state space cannot be shown to exist (see Duffie et al. (1994)).

For reasons that will become clear in the error analysis below, we need a slightly extended state space. For a small $\eta > 0$, define the state space

$$\Psi^\eta = \left\{ \theta \in \mathbb{R}^{IJ} : \sum_{i \in \mathcal{I}} \theta^i_j = 1, \theta^i_j \geq -\eta \text{ for all } i \in \mathcal{I}, j \in \mathcal{J} \right\}$$

and let $\rho = ((\rho_{\theta^i})_{i \in \mathcal{I}}, \rho_q, (\rho_{c^i}, \rho_{m^i})_{i \in \mathcal{I}})$ denote the policy function for a recursive ϵ-equilibrium. For sufficiently small η, recursive ϵ-equilibria exist (see Kubler and Schmedders (2003)). We denote the equilibrium set by $\mathcal{F}$.

6.2. *Error Analysis*

We must perturb marginal utility in order to set $h^1 = 0$ for a given ϵ-equilibrium process. However, because agents are infinitely lived, the main problem in the error analysis is that the necessary perturbations to correct for errors in $(h_{\inf} 1)$ along the event tree may propagate without bounds. Figure 3 illustrates this problem for the economy in Example 2. The asset holdings of the ϵ-equilibrium process diverge from the exact equilibrium. Setting the errors in the equilibrium equations to zero then requires increasingly larger deviations of marginal utilities. The backward substitution approach of the previous section does not yield small bounds on the necessary perturbations, and we are no longer able to show that the ϵ-equilibrium is path-approximating. Instead we show that the ϵ-equilibrium weakly approximates an economy in equilibrium. Intuitively, a weak approximation only demands that there is an exact equilibrium that can be approximately generated by the computed policy function of the recursive ϵ-equilibrium. Contrary to path-approximation, this concept allows for perturbations in the state variable. Figure 3 displays the basic idea for such an error analysis. At a perturbed value of the state variable, the approximate policy function yields the exact equilibrium variables for the current period. (The exactness in the figure is an idealization: in general, the perturbation of state variables will not yield an exact solution to the equilibrium equations, and so some additional forward and backward perturbations are necessary.) The advantage of the evaluation of the policy function at a close-by value is that the magnitude of all perturbations remains small and tight bounds for a weak approximation can be established. We now formalize the depicted intuition for the Lucas model.

To simplify the analysis, we assume that for the ϵ-equilibrium, $\mathcal{F}$, $(h_{\inf}3)$–$(h_{\inf}6)$ actually hold with equality, so for all values in $\mathcal{F}$ market clearing and budget feasibility are built in. As in the OLG model, actual errors in these equations will be negligible.

To construct a bound δ that ensures a weakly approximating ϵ-equilibrium, we show the stronger result that there exists an exact equilibrium $(\hat{z}(\sigma))_{\sigma\in\Sigma}$, such that for all $s^t \in \Sigma$, there exist some $z \in \mathcal{F}_{s_t}$ with $\|z - \hat{z}(s^t)\| < \delta$, which differs from $\hat{z}$ only in m and θ_-. No other variables are perturbed, and so $\hat{\theta}(s^t) = \theta$, $\hat{q}(s^t) = q$, $\hat{c}(s^t) = c$. For the budget constraint $(h_{\inf})$ to hold for the exact equilibrium value $\hat{z}(s^t)$, one must perturb endowments, that is, allow w^e to be nonzero. This perturbation is necessitated by the fact that for the ϵ-equilibrium values the budget constraint is assumed to hold exactly, and so perturbations in θ_- introduce an error in this equation. For these errors to be less than δ, we need to require that $\|(\hat{\theta}^i_-(s^t) - \theta_-)(q + d(s_t))\| < \delta$. As in Example 2, the strategy is to find perturbations in the endogenous state that ensure the value of the approximate policy function almost matches the exact equilibrium value.

A sufficient condition for the existence of a close-by economy in equilibrium is then that for every $\bar{z}^P$ that is identical to some $\bar{z} \in \mathcal{F}_{\bar{s}}$ in all coordinates with the exception of $\bar{m}$ and $\bar{\theta}_-$ where it has the property that $\|\bar{m}^P - \bar{m}\| < \delta$ and $\|(\bar{\theta}^P_- - \bar{\theta}_-)(\bar{q} + d(\bar{s}))\| < \delta$, we can find consistent values of next period's endogenous variables, which are identical to elements of $z(s) \in \mathcal{F}_s$, $s \in \mathcal{S}$, except that again the m's and θ_-'s satisfy the above condition. More formally, suppose that for all $\bar{s}, \bar{z} \in \mathcal{F}_{\bar{s}}$ and all $\bar{z}^P$ with $\|\bar{z}^P - \bar{z}\| < \delta$ and $\bar{q}^P = \bar{q}$, $\bar{\theta}^P = \bar{\theta}$, $\bar{c}^P = \bar{c}$, $\|(\bar{q} + d(\bar{s}))(\bar{\theta}_- - \bar{\theta}^P_-)\| < \delta$ there exist $\tilde{z}(1), \dots, \tilde{z}(S)$ such that:

(I1) For all $s \in \mathcal{S}$ it holds that $\|z(s) - \tilde{z}(s)\| < \delta$ for some $z(s) \in \mathcal{F}_s$ with

$$\tilde{q}(s) = q(s), \quad \tilde{\theta}(s) = \theta(s), \quad \tilde{c}(s) = c(s),$$
$$\big\|(q(s) + d(s))(\tilde{\theta}_-(s) - \theta_-(s))\big\| < \delta.$$

(I2) For some $\kappa \in \mathcal{K}$ and w with $\|w\| < \delta$, $h(\bar{s}, \bar{z}^P, w, \kappa, \tilde{z}(1), \dots \tilde{z}(S)) = 0$.

Then there exists an economy in equilibrium $(\mathcal{E}((w(\sigma))_{\sigma\in\Sigma}), (\hat{z}(\sigma))_{\sigma\in\Sigma})$, with $\sup_{\sigma\in\Sigma}\|w(\sigma)\| < \delta$ such that for all $s^t \in \Sigma$ and all $\hat{z}(s^t)$ it holds that $\|z - \hat{z}(s^t)\| < \delta$ for some $z \in \mathcal{F}_{s_t}$.

The rather technical conditions (I1) and (I2) nicely correspond to our intuition of a sensible error analysis for the infinite-horizon model. Given a previously perturbed point $\bar{z}^P$, we can find a perturbation $\tilde{z}$ of some point z in the equilibrium set $\mathcal{F}$ that sets the equilibrium equations at $(\bar{s}, \bar{z}^P)$ equal to zero for a slightly perturbed ($\|w\| < \delta$) economy. An iterated application of this process then leads to an exact equilibrium process $\hat{z}$ for a close-by economy.

To verify the conditions for a given ϵ-equilibrium, we need some additional notation. For an under determined system $Ax = b$ with a matrix A that has linearly independent rows, denote by $A^+ = A^\top(AA^\top)^{-1}$ the pseudo inverse of A. The unique solution of the system that minimizes the Euclidean norm $\|x\|_2$ is then given by $x_{\mathrm{LS}} = A^+b$. We use the Euclidean norm here since it is well understood how to compute A^+b accurately. The approach immediately yields an upper bound on the sup-norm of the error since $\|x\| \le \|x\|_2$ for $x \in \mathbb{R}^n$. Using the fact that we consider a recursive ϵ-equilibrium, we can write current endogenous variables as functions of θ_- alone and define for any current shock $\bar{s}$ and any $\theta_-(1), \ldots, \theta_-(S) \in \Psi^\eta$ a $J \times S$ payoff matrix by

$$\begin{aligned} M(\bar{s}, \vec{\theta}_-) &= M\big(\bar{s}, \theta_-(1), \ldots, \theta_-(S)\big) \\ &= \big(\beta\Pi(s|\bar{s})\big(\rho_{q_j}(s, \theta_-(s)) + d_j(s)\big)\big)_{js}. \end{aligned}$$

For any $\bar{m}^P$, $\bar{s}$, $\bar{z} \in \mathcal{F}_{\bar{s}}$, $\kappa \in \mathbb{R}^J_+$, and $\vec{\theta}_-$, define S-dimensional vectors for each agent i:

$$\begin{aligned} &E^i(\bar{s}, \bar{m}^P, \bar{q}, \kappa, \vec{\theta}_-) \\ &\quad = (M(\bar{s}, \vec{\theta}_-))^+ \left[\bar{q}\bar{m}^{Pi} - M(\bar{s}, \vec{\theta}_-)\begin{pmatrix} u_i'(\rho_{c^i}(1, \theta_-(1))) \\ \vdots \\ u_i'(\rho_{c^i}(S, \theta_-(S))) \end{pmatrix} - \kappa\right]. \end{aligned}$$

These are the necessary perturbations in $m^i(1), \ldots, m^i(S)$ for $(h_{\inf}1)$ to hold with equality if the next period's states are $\vec{\theta}$ and this period's endogenous variables are identical to $\bar{z}$, except that this period's marginal utilities might differ from $\bar{m}$ and be given by $\bar{m}^P$. The formula appears complicated only because we consider the general case of J assets and S states. In the example below it simplifies considerably. Lemma 2 gives sufficient conditions to ensure that (I1) and (I2) hold.

LEMMA 2: *Given $\delta > 0$, suppose that for all $\bar{s} \in \mathcal{S}$, $\bar{z} \in \mathcal{F}_{\bar{s}}$, and all $(n_1, \ldots, n_I) \in \{-1, 1\}^I$, there exists $\vec{\theta}_- \in (\Psi^\eta)^S$ with* $\max_{i\in\mathcal{I}, s\in\mathcal{S}} \|(\theta^i_-(s) - \bar{\theta}^i)(\rho_q(s, \theta_-(s)) +$

$d(s))\| < \delta$ such that for all $i \in \mathcal{I}$ there exists a $\kappa \geq 0$ with $\kappa\bar{\theta}^i = 0$, which ensures that for all $s \in \mathcal{S}$,

$$|E_s^i(\bar{s}, \bar{m}, \bar{q}, \kappa, \vec{\theta}_-)| < \min\big(\delta, \delta u_i'\big(\rho_{c^i}(s, \theta_-(s))\big)\big) \quad \textit{and}$$

$$\operatorname{sgn}(E_s^i(\bar{s}, \bar{m}, \bar{q}, \kappa, \vec{\theta}_-)) = n_i.$$

Then there exists an economy in equilibrium $(\mathcal{E}((w(\sigma))_{\sigma\in\Sigma}), (\hat{z}(\sigma))_{\sigma\in\Sigma})$, with $\sup_{\sigma\in\Sigma}\|w(\sigma)\| < \delta$ such that for all $s^t \in \Sigma$ and all $\hat{z}(s^t)$ it holds that $\|z - \hat{z}(s^t)\| < \delta$ for some $z \in \mathcal{F}_{s_t}$.

For all $\bar{\theta}_- \in \Psi^\eta$ and all $\bar{s} \in \mathcal{S}$ with associated $\bar{q}, \bar{m}, \bar{\theta}$, one can perform a grid search to verify the conditions of the lemma and to determine a bound on backward errors in endowments, $w_{\max}^e$ as well as errors in utilities, $w_{\max}^u$. We illustrate below some problems that arise in practice.

If one wants to perturb endowments only, one must translate the perturbations in marginal utility into perturbations in consumption values and perturb individual endowments to satisfy the budget constraints. A crude bound can be obtained by computing

$$\max_{s\in\mathcal{S}, i\in\mathcal{I}, \theta\in\Theta}\big\{u_i'^{-1}\big(u_i'(\rho_{c^i}(s, \theta)) + w_{\max}^u\big) - \rho_{c^i}(s, \theta), u_i'^{-1}\big(u_i'(\rho_{c^i}(s, \theta)) - w_{\max}^u\big) - \rho_{c^i}(s, \theta)\big\}.$$

The total necessary perturbation is given by the sum of this expression and $w_{\max}^e$. In Table II we denote this by $\bar{w}_{\max}^e$.

6.3. *Parametric Example*

The following small example illustrates the analysis above. There are $S = 2$ shocks, which are i.i.d. and equiprobable. There are two agents with CRRA utility functions that have identical coefficient of risk aversion of $\gamma = 1.5$ who discount the future with $\beta = 0.95$. Individual endowments are $e^1 = (1.5, 3.5)$ and $e^2 = (3.5, 1.5)$. There is a single tree with dividends $d(s) = 1$ for $s = 1, 2$.

Since there are only two agents, the endogenous state space for the recursive ϵ-equilibrium simply consists of the interval [0, 1]. For our error analysis it is crucial, however, that we can perturb $\theta_-(s)$ even if this period's choice is $\bar{\theta} = 0$.

TABLE II
ERRORS

ϵ	$w_{\max}^u$	$w_{\max}^e$	$\bar{w}_{\max}^e$
2.04 (−3)	3.53 (−3)	1.19 (−3)	4.2 (−3)

For this, we extend the state space to $[-0.01, 1.01]$. The bounds should be chosen to guarantee that nonnegative consumption is still feasible at all points in the state space. Standard algorithms (see, e.g., Kubler and Schmedders (2003)) can be used to compute a recursive ϵ-equilibrium even for this extended state space. At $\bar{\theta}_- < 0$, the short-sale constraint forces the new choice $\rho_\theta(\bar{s}, \bar{\theta})$ to be nonnegative, no matter what the current shock, and our error analysis as outlined above goes through. We obtain the errors reported in Table II.

7. CONCLUSION

An error analysis should ideally relate the computed solution to an exact equilibrium of the underlying economy. However, unfortunately this is generally impossible. Instead, we argue that it is often economically interesting to relate the computed solution to an exact solution of a close-by economy and to perform a backward error analysis.

For stochastic infinite-horizon models we define economies to be close by if preferences and endowments are close by in the sup-norm. With this definition, we show how to construct ϵ-equilibrium processes from computed candidate solutions that approximate an exact equilibrium of a close-by economy. When agents are finitely lived, this construction is straightforward, as the process is generated by the transition function of the recursive approximate equilibrium. When agents are infinitely lived, the construction is more elaborate, since one cannot guarantee sample-path stability of the equilibrium transition. In practice one needs to be content with a notion of "weak approximation."

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APPENDIX

PROOF OF LEMMA 1: Define $z \in \ell_\infty$ by $z_l(s^t) = \delta_l(s^t)x_l(s^t)$ for all l, s^t. For $y \in \ell_\infty$, define $t(y) = \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi(s^t)(D_x u(x(s^t), s_t) + w(s^t)) \cdot y(s^t)$. The closed half-space $\{y : t(y) \geq t(x)\}$ does not contain any point from the interior of $\mathcal{B}$. Postlewaite and Schmeidler (1981, p. 109) construct an indifference curve that is identical to the indifference curve of U passing through x in this half-space and that is identical to the boundary of the half-space in the region where the indifference curve of U is outside of the half-space. By their construction, to prove the theorem, it suffices to derive an upper bound on

$$\epsilon = d^H\big(\{y : U(y) \geq U(x)\}, \big\{y : t(y) \geq t(x) \wedge U(y) \geq U(x)\big\}\big).$$

Observe that the maximum will be obtained at some $\bar{y}$ satisfying $U(\bar{y}) = U(x)$ and $D_x U(\bar{y})$ collinear to $D_x U(x) + w$. By the definition of d^H, we must have $\|\bar{y} - x\| \geq \epsilon$. By homotheticity, there exists a scalar $\alpha > 0$ such that $\bar{y} = \alpha z$. Since monotonicity of preferences implies that $\bar{y}_l(s^t) \geq x_l(s^t)$ for some l, s^t, $\|\bar{y} - x\|$ must be bounded by $\|\bar{z} - x\|$, where $\bar{z} = z / \sup_\sigma \|\delta(\sigma)\|$. This is true because $\bar{z} = \bar{\alpha} z$ for some $\bar{\alpha}$ and $\bar{z} \leq \bar{y}$. By definition of z it follows that $\epsilon \leq \sup_{s^t, l} x_l(s^t)(1 - \frac{\delta_l(s^t)}{\sup_\sigma \|\delta(\sigma)\|})$. *Q.E.D.*

PROOF OF LEMMA 2: To prove the lemma, it is useful to consider relative errors as opposed to absolute errors and to define $R^i(\bar{s}, \bar{m}, \bar{q}, \kappa, \vec{\theta}_-)$ by $R^i_s = E^i_s / u'_i(\rho_{c^i}(s, \theta_-(s)))$ for all $s \in \mathcal{S}$. With this definition,

$$\bar{q}\bar{m}^{Pi} - M(\bar{s}, \vec{\theta}_-) \begin{pmatrix} (1 + R^i_1) u'_i(\rho_{c^i}(1, \theta_-(1))) \\ \vdots \\ (1 + R^i_S) u'_i(\rho_{c^i}(S, \theta_-(S))) \end{pmatrix} - \kappa = 0.$$

It follows that for any scalar $\gamma > -1$, there is some $\kappa' \geq 0$, $\kappa' \bar{\theta}^i = 0$ such that for any $s \in \mathcal{S}$,

$$\left| R^i_s\big(\bar{s}, (1+\gamma)\bar{m}^P, \bar{q}, \kappa', \vec{\theta}\big) \right| \leq \left| (1+\gamma)\big(1 + R^i_s(\bar{s}, \bar{m}^P, \bar{q}, \kappa, \vec{\theta}_-)\big) - 1 \right|. \tag{1}$$

Given $\bar{s}, \bar{z} \in \mathcal{F}_{\bar{s}}$ and a perturbed $\bar{z}^P$ with $\bar{q}^P = \bar{q}$, $\|\bar{m}^P - \bar{m}\| < \delta$, we can write $\bar{m}^{iP} = (1 + \gamma_i)\bar{m}^i$ for some $\gamma = (\gamma_1, \dots, \gamma_I)$ with $\|\gamma\| \leq \delta$. Given $(\operatorname{sgn}(\gamma_1), \dots, \operatorname{sgn}(\gamma_I)) \in \{-1, 1\}^I$, the conditions of the lemma require that there exists $\vec{\theta}_-$ such that for all $i \in \mathcal{I}$, we can find κ such that

$$|R^i_s(\bar{s}, \bar{m}, \bar{q}, \kappa, \vec{\theta}_-)| \leq \min\left(\delta, \frac{\delta}{u'_i(\rho_{c^i}(s, \theta_-(s)))}\right) \quad \text{and}$$

$$\operatorname{sgn}(R^i_s(\bar{s}, \bar{m}, \bar{q}, \kappa, \vec{\theta}_-)) = -\operatorname{sgn}(\gamma_i).$$

With (1), because of the sign condition, we obtain $|R^i_s(\bar{s}, \bar{m}(1+\gamma_i), \bar{q}, \kappa, \vec{\theta}_-)| \leq \min(\delta, \frac{\delta}{u'_i(\rho_{c^i}(s, \theta_-(s)))})$ for all $s \in \mathcal{S}$. This shows that errors do not propagate and therefore proves the lemma. *Q.E.D.*

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